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http://www1.unex.es/eweb/fisteor/santos/sby.html

Weighted average finite difference methods for fractional diffusion equations

The fractional version of the Crank-Nicolson method is just a particular case of the weighted average method in which the weight parameter λ is 1/2

Here I provide a *Mathematica* notebook that uses the weighted-averaged difference method for solving some class of fractional diffusion equations. The method is discussed in

Ref[1] = S. B. Yuste, *Weighted average finite difference methods for fractional diffusion equations*, Proceedings of FDA'04, pages 335-340 (2004) [cs.NA/0408053]

and

Ref [2] = S. B. Yuste, *Weighted average finite difference methods for fractional diffusion equations*, Journal of Computational Physics 216 (2006) 264-274

The fractional diffusion equation (FDE) we are going to solve is (see Eq. (1) of Ref[2]) :

$$\frac{\partial u}{\partial t} = K \ {}_0D^{1-\gamma} \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

In particular, as an example, we are going to use the initial condition u(x,0)=x(1-x) and the Dirichlet boundary conditions u(0,t)=u(1,t)=0

The solution u evaluated at position x_j at time t_m, $u(x_j,t_m)$, is denoted here as u[j,m,γ], where the third index γ is the anomalous diffusion exponent of the FDE.

The exact solution

uExactx1mx[x,t,γ,Kg,*numterm*] is the exact solution of the FDE (1) written as a (finite) series with *numterm* terms (the exact solution corresponds to *numterm*→∞)

```
uExactx1mx[x_, t_, γ_, Kg_, numterm_] :=
 8 / Pi^3 * Sum[1 / (2 * n + 1) ^3 * Sin[(2 * n + 1) * Pi * x] *
    MittagLefflerE[γ, -Kg * (2 * n + 1) ^2 * Pi^2 * t^γ], {n, 0, numterm}]
```

(The Mittag-Leffler function for γ = 1/2 can be written in a simpler way: MittagLefflerE[1/2,z]=e^{z^2} Erfc[-z])

The Weighted averaged (WA) numerical algorithm (see Eq.(19) of Ref[2]):

BDF1 Coefficients: w[k_, alfa_]≡$\omega_k^{(\alpha)}$ (see formula (13) of Ref[2]) :

```
w[0, alfa_] := 1;
w[k_, alfa_] := w[k, alfa] = (1 - (alfa + 1) / k) * w[k - 1, alfa]
```

In what follows,

lambda≡λ (weight factor; see Eq. (17) of [2]) ,

sfrac≡S (parameter defined in Eq. (19b) of [2]) and

Nx determines the space discretization of the interval in which the equation is going to be solve (the total number of points x_j is 2Nx+1)

```
RR[j_, m_, gamma_] := Module[{kk},
  kk = Sum[((1 - lambda) * w[k + 1, 1 - gamma] + lambda * w[k, 1 - gamma]) *
     (uWA[j - 1, m - 1 - k, gamma] - 2 * uWA[j, m - 1 - k, gamma] +
       uWA[j + 1, m - 1 - k, gamma]), {k, 0, m - 1}];
  uWA[j, m - 1, gamma] + sfrac * kk]
```

uWA[j, m, gamma] provides the value of the numeral solution at the point x_j at time t_m when the anomalous diffusion exponent γ is gamma.

```
uWALista[m_, gamma_] :=
 uWALista[m, gamma] = LinearSolve[SparseArray[{Band[{1, 2}] → Table[
        Which[j == Nx - 1, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)], {j, -Nx, Nx - 1}],
      Band[{1, 1}] → Table[Which[j == -Nx, 1, j == Nx, 1, True,
          (1 + 2 * sfrac * w[0, 1 - gamma] * (1 - lambda))], {j, -Nx, Nx}],
      Band[{2, 1}] → Table[Which[j == -Nx, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)],
        {j, -Nx, Nx - 1}]}, 2 * Nx + 1], Table[
    Which[j == -Nx, uL[m * ht], j == Nx, uR[m * ht], True, RR[j, m, gamma]], {j, -Nx, Nx}]]

uWA[j_, m_, gamma_] := uWA[j, m, gamma] = uWALista[m, gamma][[Nx + 1 + j]]
```

soluWA[t, gamma] is a function that generates a table with the solution u evaluated at time t. The table has the format {x, u (x, t)}

```
soluWA[t_, gamma_] := Module[{mfin, kk, uMEfinalTab, eps = $MachineEpsilon},
  mfin = Floor[(t + eps) / ht];
  kk = Timing[Do[uWAfinalTab = Table[
         {(Nx + j) * deltax, uWA[j, m, gamma]}, {j, -Nx, Nx}], {m, 0, mfin}]][[1]];
  Print["t=", mfin * ht, " : mfin=", mfin, " : ht=", ht, " : Delta x=",
   deltax, " : Weight factor λ=", lambda,
   " : Computing time=", kk, " : Weighted-Averaged method"];
  uWAfinalTab]
```

Examples

Numerical solution of the FDE by means of the weighted-averaged method with weight factor λ=1/2, i.e., by means of the fractional Crank-Nicolson method, for

γ=1/2, K=1, , sfrac=0.7 , Δx=1/20 , Δt=6.25×10^{-6}.

The fractional Crank-Nicolson method is always stable

Here we clear the saved values of uWALista and uWA (the evaluation of this cell is required if you are going to work with several examples in a Mathematica session)

```
Clear[uWALista, uWA];
uWALista[m_, gamma_] :=
 uWALista[m, gamma] = LinearSolve[SparseArray[{Band[{1, 2}] → Table[
       Which[j == Nx - 1, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)], {j, -Nx, Nx - 1}],
     Band[{1, 1}] → Table[Which[j == -Nx, 1, j == Nx, 1, True,
        (1 + 2 * sfrac * w[0, 1 - gamma] * (1 - lambda))], {j, -Nx, Nx}],
     Band[{2, 1}] → Table[Which[j == -Nx, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)],
       {j, -Nx, Nx - 1}]}, 2 * Nx + 1],
   Table[Which[j == -Nx, uL[m * ht], j == Nx, uR[m * ht], True, RR[j, m, gamma]],
    {j, -Nx, Nx}]]
uWA[j_, m_, gamma_] := uWA[j, m, gamma] = uWALista[m, gamma][[Nx + 1 + j]]
```

lambda=weight factor=λ

```
lambda = 0.5;
```

2*Nx+1=number of spatial points

```
Nx = 10;
```

Parameters of this example: gamma = 1/2, Kg = 1, sfrac = 1, deltax = 1./(2*Nx)

(Notation: gamma=γ, ht=Δt , Kg=K, sfrac=S, deltax=Δx)

```
{gamma = 0.5, deltax = 1. / (2 * Nx), Kg = 1.;
 sfrac = 1, ht = (sfrac * deltax^2 / Kg) ^ (1 / gamma)}
```

$\{0.5, 0.05, 1, 6.25\times10^{-6}\}$

According to Eq. (27) of Ref[2], the critical stability value for sfrac≡S=K$(\Delta t)^{\gamma}/(\Delta x)^2$ is given by

$S_\times$ = 1/(2*(2*lambda-1)* w(-1,1-gamma)

As we use the BDF1 coefficients, we have

w(-1,1-gamma)=2^(1-gamma)

The present WA numerical method is stable if

$1 / \text{sfracEfect} \geq 1 / S_\times$

with

sfracEfect = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2

(see Eq. (25) of Ref[2])

```
{(Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2 // N;
 sfracEfectivo = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2;
 scritico = 1 / (2 * (2 * lambda - 1) * 2^(1 - gamma));
 1 / sfracEfectivo, 1 / scritico}
```

... Power: Infinite expression $\frac{1}{0.}$ encountered.

$\{\text{Sec}[\frac{\pi}{40}]^2, 0\}$

Here we see that $1 / \text{sfracEfect} = \text{Sec}[\frac{\pi}{40}]^2$ is larger than $1 / S_\times = 0$. Therefore, the algorithm is stable

We set the initial condition : u(x,t=0)= x(1-x)

```
uWA[nx_, 0, alfa_] := (nx + Nx) * deltax * (1 - (nx + Nx) * deltax)
```

```
Table[{nx, uWA[nx, 0, 1]}, {nx, -Nx, Nx}] // N
```

```
{{-10., 0.}, {-9., 0.0475}, {-8., 0.09}, {-7., 0.1275}, {-6., 0.16},
 {-5., 0.1875}, {-4., 0.21}, {-3., 0.2275}, {-2., 0.24}, {-1., 0.2475},
 {0., 0.25}, {1., 0.2475}, {2., 0.24}, {3., 0.2275}, {4., 0.21},
 {5., 0.1875}, {6., 0.16}, {7., 0.1275}, {8., 0.09}, {9., 0.0475}, {10., 0.}}
```

Boundary condition: u(0,t)=u(1,t)=0.

```
uL[t_] := 0;
uR[t_] := 0;
```

```
uWA[-Nx, m_, alfa_] := uL[m * ht];
uWA[Nx, m_, alfa_] := uR[m * ht];
```

The solution at the points x_j after mfin= 2 timesteps.

```
mfin = 2; uWALista[mfin, gamma]
```

```
{0.0220118, 0.0440236, 0.0849551, 0.121864, 0.15416, 0.181593, 0.204072,
 0.221565, 0.234063, 0.241563, 0.244063, 0.241563, 0.234063, 0.221565,
 0.204072, 0.181593, 0.15416, 0.121864, 0.0849551, 0.0440236, 0.0220118}
```

The solution in the format {....{x_j, u(x_j, t_m)}, {x_{j+1}, u(x_{j+1}, t_m)},....} where m≡mfin

```
mfin = 1; soluWA[mfin * ht, gamma]
```

t=6.25×10^{-6} : mfin=1 : ht=6.25×10^{-6} : Delta x=0.05
: Weight factor λ=0.5 : Computing time=0. : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.0447548}, {0.1, 0.0865192}, {0.15, 0.123822},
 {0.2, 0.156269}, {0.25, 0.183755}, {0.3, 0.206251}, {0.35, 0.22375},
 {0.4, 0.23625}, {0.45, 0.24375}, {0.5, 0.24625}, {0.55, 0.24375},
 {0.6, 0.23625}, {0.65, 0.22375}, {0.7, 0.206251}, {0.75, 0.183755},
 {0.8, 0.156269}, {0.85, 0.123822}, {0.9, 0.0865192}, {0.95, 0.0447548}, {1., 0}}
```

```
mfin = 10; soluWA[mfin * ht, gamma]
```

t=0.0000625 : mfin=10 : ht=6.25×10^{-6} : Delta x=0.05
: Weight factor λ=0.5 : Computing time=0.015625 : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.0407488}, {0.1, 0.0793207}, {0.15, 0.114589},
 {0.2, 0.145849}, {0.25, 0.172675}, {0.3, 0.194816}, {0.35, 0.21213},
 {0.4, 0.224537}, {0.45, 0.231995}, {0.5, 0.234482}, {0.55, 0.231995},
 {0.6, 0.224537}, {0.65, 0.21213}, {0.7, 0.194816}, {0.75, 0.172675},
 {0.8, 0.145849}, {0.85, 0.114589}, {0.9, 0.0793207}, {0.95, 0.0407488}, {1., 0}}
```

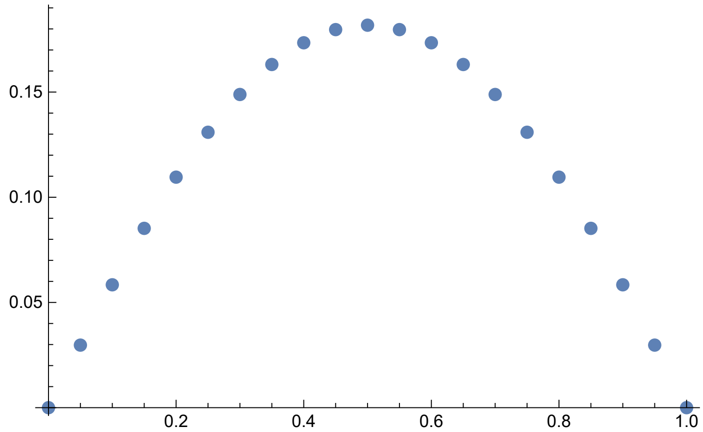
```
mfin = 100; soluWA[mfin * ht, gamma]
```

```
t=0.000625 : mfin=100 : ht=6.25×10^-6 : Delta x=0.05
  : Weight factor λ=0.5 : Computing time=0.640625 : Weighted-Averaged method
```

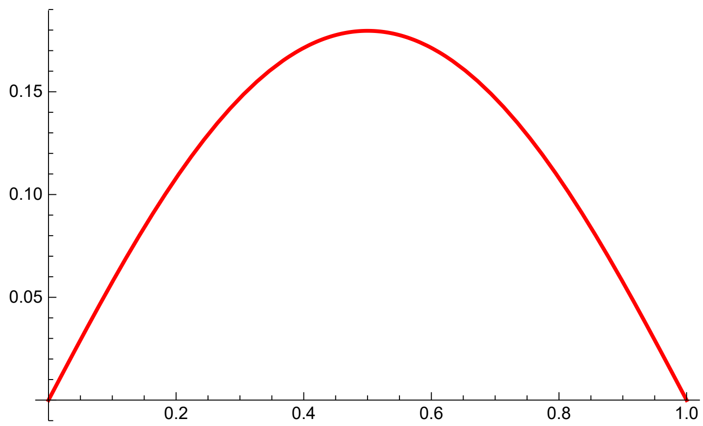
```
{{0., 0}, {0.05, 0.0329848}, {0.1, 0.0647021}, {0.15, 0.0942626},
 {0.2, 0.120974}, {0.25, 0.144305}, {0.3, 0.163856}, {0.35, 0.179331},
 {0.4, 0.190523}, {0.45, 0.19729}, {0.5, 0.199554}, {0.55, 0.19729},
 {0.6, 0.190523}, {0.65, 0.179331}, {0.7, 0.163856}, {0.75, 0.144305},
 {0.8, 0.120974}, {0.85, 0.0942626}, {0.9, 0.0647021}, {0.95, 0.0329848}, {1., 0}}
```

```
mfin = 200;
figNum = ListPlot[soluWA[mfin * ht, gamma], PlotRange → All, PlotStyle → PointSize[0.02]]
```

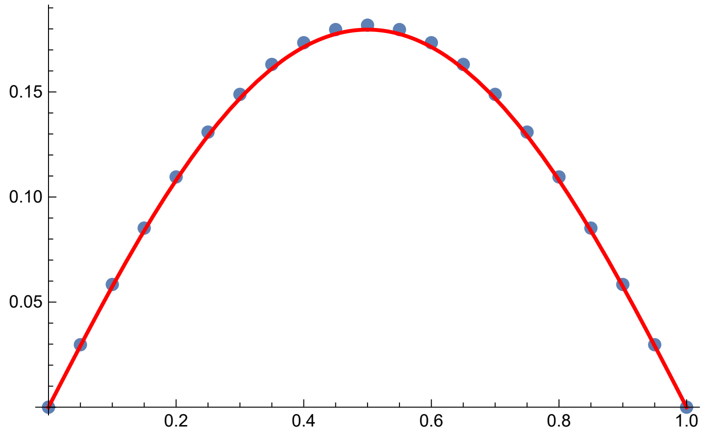
```
t=0.00125 : mfin=200 : ht=6.25×10^-6 : Delta x=0.05
  : Weight factor λ=0.5 : Computing time=1.67188 : Weighted-Averaged method
```




```
tv = 200 * ht;
figExact = Plot[uExactx1mx[x, tv, 1 / 2, 1, 20],
  {x, 0, 1}, PlotStyle → {Red, Thick}, PlotRange → All]
```



```
Show[figNum, figExact, PlotRange → All]
```



Numerical solution of the FDE by means of the weighted-averaged method with weight factor λ=1/2, i.e., by means of the fractional Crank-Nicolson method, for

γ=3/4, K=1, , sfrac=0.7 , Δx=1/20 , Δt=6.25×10^{-6}.

The fractional Crank-Nicolson method is always stable

Here we clear the saved values of uWALista and uWA (the evaluation of this cell is required if you are going to work with several examples in a Mathematica session)

```
Clear[uWALista, uWA];
uWALista[m_, gamma_] :=
 uWALista[m, gamma] = LinearSolve[SparseArray[{Band[{1, 2}] → Table[
        Which[j == Nx - 1, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)], {j, -Nx, Nx - 1}],
      Band[{1, 1}] → Table[Which[j == -Nx, 1, j == Nx, 1, True,
          (1 + 2 * sfrac * w[0, 1 - gamma] * (1 - lambda))], {j, -Nx, Nx}],
      Band[{2, 1}] → Table[Which[j == -Nx, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)],
        {j, -Nx, Nx - 1}]}, 2 * Nx + 1],
    Table[Which[j == -Nx, uL[m * ht], j == Nx, uR[m * ht], True, RR[j, m, gamma]],
     {j, -Nx, Nx}]]
uWA[j_, m_, gamma_] := uWA[j, m, gamma] = uWALista[m, gamma][[Nx + 1 + j]]
```

lambda=weight factor=λ

```
lambda = 0.5;
```

2*Nx+1=number of spatial points

```
Nx = 10;
```

Parameters of this example: gamma = 3/4, Kg = 1, sfrac = 1, deltax = 1./(2*Nx)

(Notation: gamma=γ, ht=Δt , Kg=K, sfrac=S, deltax=Δx)

```
{gamma = 0.75, deltax = 1. / (2 * Nx), Kg = 1.;
 sfrac = 1, ht = (sfrac * deltax^2 / Kg) ^ (1 / gamma)}
{0.75, 0.05, 1, 0.000339302}
```

According to Eq. (27) of Ref[2], the critical stability value for sfrac≡S=K(Δt)$^{\gamma}$/(Δx)^2 is given by

$S_\times$ = 1/(2*(2*lambda-1)* w(-1,1-gamma)

As we use the BDF1 coefficients, we have w(-1,1-gamma)=2^(1-gamma)

The present WA numerical method is stable if

1 / sfracEffect ≥ 1 / S_x

with

sfracEffect = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2

(see Eq. (25) of Ref[2])

```
{(Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2 // N;
 sfracEfectivo = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2;
 scritico = 1 / (2 * (2 * lambda - 1) * 2^(1 - gamma));
 1 / sfracEfectivo, 1 / scritico}
```

... Power: Infinite expression $\frac{1}{0.}$ encountered.

{$\text{Sec}\left[\frac{\pi}{40}\right]^2$, 0}

Here we see that 1 / sfracEffect = $\text{Sec}\left[\frac{\pi}{40}\right]^2$ is larger than 1 / S_x = 0. Therefore, the algorithm is stable

We set the initial condition : u(x,t=0)= x(1-x)

```
uWA[nx_, 0, alfa_] := (nx + Nx) * deltax * (1 - (nx + Nx) * deltax)
```

```
Table[{nx, uWA[nx, 0, 1]}, {nx, -Nx, Nx}] // N
```

{{-10., 0.}, {-9., 0.0475}, {-8., 0.09}, {-7., 0.1275}, {-6., 0.16}, {-5., 0.1875}, {-4., 0.21}, {-3., 0.2275}, {-2., 0.24}, {-1., 0.2475}, {0., 0.25}, {1., 0.2475}, {2., 0.24}, {3., 0.2275}, {4., 0.21}, {5., 0.1875}, {6., 0.16}, {7., 0.1275}, {8., 0.09}, {9., 0.0475}, {10., 0.}}

Boundary condition: u(0,t)=u(1,t)=0.

```
uL[t_] := 0;
uR[t_] := 0;
```

```
uWA[-Nx, m_, alfa_] := uL[m * ht];
uWA[Nx, m_, alfa_] := uR[m * ht];
```

The solution at the points x_j after mfin= 2 timesteps.

```
mfin = 2; uWALista[mfin, gamma]
```

{0.021454, 0.0429081, 0.0833106, 0.120016, 0.152241, 0.179651, 0.202122, 0.219613, 0.232111, 0.23961, 0.24211, 0.23961, 0.232111, 0.219613, 0.202122, 0.179651, 0.152241, 0.120016, 0.0833106, 0.0429081, 0.021454}

The solution in the format {....{x_j, u(x_j, t_m)}, {x_{j+1}, u(x_{j+1}, t_m)},....} where m≡mfin

```
mfin = 1; soluWA[mfin * ht, gamma]
```

```
t=0.000339302 : mfin=1 : ht=0.000339302 : Delta x=0.05
  : Weight factor λ=0.5 : Computing time=0. : Weighted-Averaged method
```

```
{{0., 0}, {0.05, 0.0442973}, {0.1, 0.0859391}, {0.15, 0.123209},
 {0.2, 0.155648}, {0.25, 0.183131}, {0.3, 0.205627}, {0.35, 0.223125},
 {0.4, 0.235625}, {0.45, 0.243125}, {0.5, 0.245625}, {0.55, 0.243125},
 {0.6, 0.235625}, {0.65, 0.223125}, {0.7, 0.205627}, {0.75, 0.183131},
 {0.8, 0.155648}, {0.85, 0.123209}, {0.9, 0.0859391}, {0.95, 0.0442973}, {1., 0}}
```

```
mfin = 10; soluWA[mfin * ht, gamma]
```

```
t=0.00339302 : mfin=10 : ht=0.000339302 : Delta x=0.05
  : Weight factor λ=0.5 : Computing time=0.015625 : Weighted-Averaged method
```

```
{{0., 0}, {0.05, 0.0369242}, {0.1, 0.0724036}, {0.15, 0.105381},
 {0.2, 0.135052}, {0.25, 0.160835}, {0.3, 0.182324}, {0.35, 0.199248},
 {0.4, 0.211436}, {0.45, 0.218784}, {0.5, 0.221239}, {0.55, 0.218784},
 {0.6, 0.211436}, {0.65, 0.199248}, {0.7, 0.182324}, {0.75, 0.160835},
 {0.8, 0.135052}, {0.85, 0.105381}, {0.9, 0.0724036}, {0.95, 0.0369242}, {1., 0}}
```

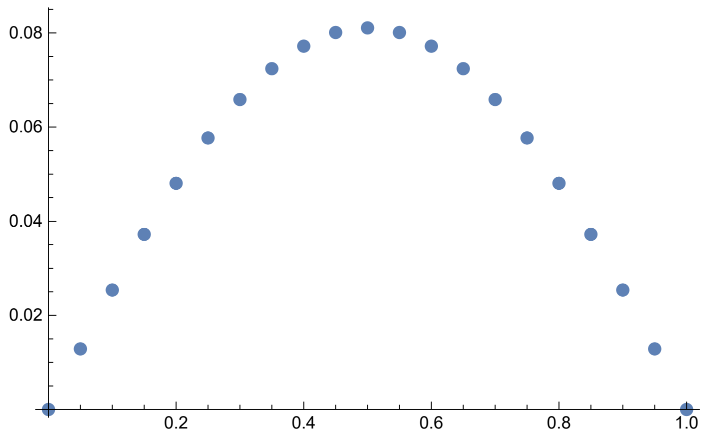
```
mfin = 100; soluWA[mfin * ht, gamma]
```

```
t=0.0339302 : mfin=100 : ht=0.000339302 : Delta x=0.05
  : Weight factor λ=0.5 : Computing time=0.59375 : Weighted-Averaged method
```

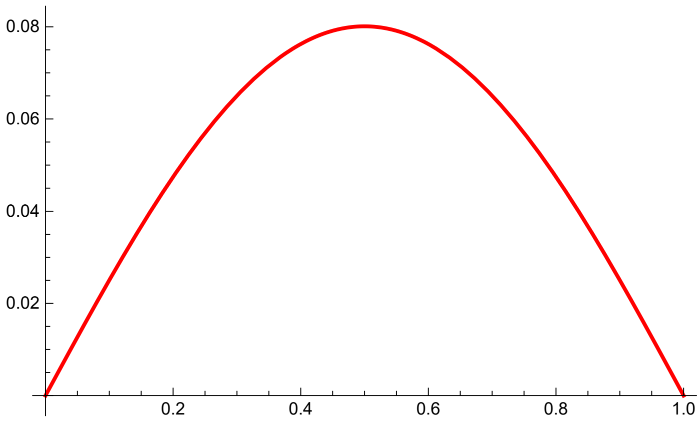
```
{{0., 0}, {0.05, 0.0194712}, {0.1, 0.0383819}, {0.15, 0.0562424},
 {0.2, 0.0726157}, {0.25, 0.0871203}, {0.3, 0.0994317}, {0.35, 0.109284},
 {0.4, 0.116469}, {0.45, 0.12084}, {0.5, 0.122306}, {0.55, 0.12084},
 {0.6, 0.116469}, {0.65, 0.109284}, {0.7, 0.0994317}, {0.75, 0.0871203},
 {0.8, 0.0726157}, {0.85, 0.0562424}, {0.9, 0.0383819}, {0.95, 0.0194712}, {1., 0}}
```

```
mfin = 200;
figNum = ListPlot[soluWA[mfin * ht, gamma], PlotRange → All, PlotStyle → PointSize[0.02]]
```

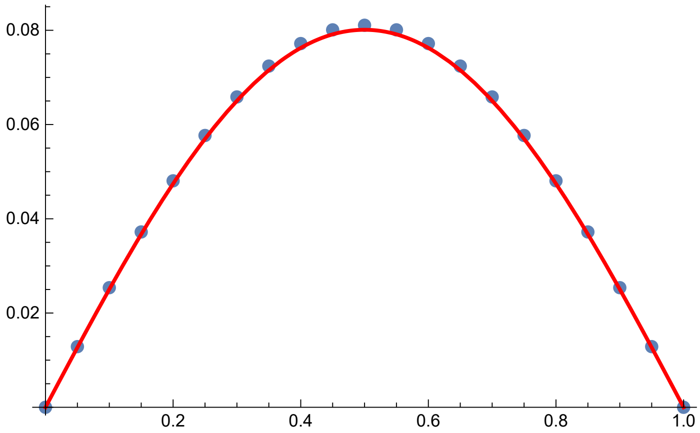
```
t=0.0678604 : mfin=200 : ht=0.000339302 : Delta x=0.05
  : Weight factor λ=0.5 : Computing time=1.57813 : Weighted-Averaged method
```



```
tv = 200 * ht;
figExact = Plot[uExactx1mx[x, tv, 3 / 4, 1, 20],
  {x, 0, 1}, PlotStyle → {Red, Thick}, PlotRange → All]
```




```
Show[figNum, figExact, PlotRange → All]
```



Numerical solution of the FDE by means of the weighted-averaged method with weight factor λ=0.8 for

γ=1/2, K=1, , sfrac=0.7 , Δx=1/20 , Δt=3.0625×10^{-6}

The algorithm is unstable in this case

Here we clear the saved values of uWALista and uWA (the evaluation of this cell is required if you are going to work with several examples in a Mathematica session)

```
Clear[uWALista, uWA];
uWALista[m_, gamma_] :=
 uWALista[m, gamma] = LinearSolve[SparseArray[{Band[{1, 2}] → Table[
        Which[j == Nx - 1, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)], {j, -Nx, Nx - 1}],
      Band[{1, 1}] → Table[Which[j == -Nx, 1, j == Nx, 1, True,
          (1 + 2 * sfrac * w[0, 1 - gamma] * (1 - lambda))], {j, -Nx, Nx}],
      Band[{2, 1}] → Table[Which[j == -Nx, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)],
        {j, -Nx, Nx - 1}]}, 2 * Nx + 1],
    Table[Which[j == -Nx, uL[m * ht], j == Nx, uR[m * ht], True, RR[j, m, gamma]],
     {j, -Nx, Nx}]]
uWA[j_, m_, gamma_] := uWA[j, m, gamma] = uWALista[m, gamma][[Nx + 1 + j]]
```

lambda=weight factor=λ

```
lambda = 0.8;
```

2*Nx+1=number of spatial points

```
Nx = 10;
```

Parameters of the example : gamma = 1/2, Kg = 1, sfrac = 0.7, deltax = 1./(2*Nx)

Notation: gamma=γ, ht=Δt , Kg=K , sfrac=S , deltax=Δx

```
{gamma = 0.5, deltax = 1. / (2 * Nx), Kg = 1.;
 sfrac = 0.7, ht = (sfrac * deltax^2 / Kg) ^ (1 / gamma)}
```

$\{0.5, 0.05, 0.7, 3.0625 \times 10^{-6}\}$

According to Eq. (27) of Ref[2], the critical stability value for sfrac≡S=K(Δt)$^{\gamma}$/(Δx)^2 is given by

$S_{\times}$ = 1/(2*(2*lambda-1)* w(-1,1-gamma)

As we use the BDF1 coefficients, we have

w(-1,1-gamma)=2^(1-gamma)

The present numerical method is stable if 1 / sfracEfect ≥ 1 / $S_{\times}$ with sfracEfect = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2 (see Eq. (25) of Ref[2])

```
{(Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2 // N;
 sfracEfectivo = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)])^2;
 scritico = 1 / (2 * (2 * lambda - 1) * 2^(1 - gamma));
 1 / sfracEfectivo, 1 / scritico}
```

{1.43742, 1.69706}

Here we see that 1 / sfracEfect = 1.43742 is smaller than 1 / $S_{\times}$ = 1.69706 . Therefore, the algorithm is unstable

Initial condition : u(x,t=0)= x(1-x)

```
uWA[nx_, 0, alfa_] := (nx + Nx) * deltax * (1 - (nx + Nx) * deltax)
```

```
Table[{nx, uWA[nx, 0, 1]}, {nx, -Nx, Nx}] // N
```

{{-10., 0.}, {-9., 0.0475}, {-8., 0.09}, {-7., 0.1275}, {-6., 0.16},
{-5., 0.1875}, {-4., 0.21}, {-3., 0.2275}, {-2., 0.24}, {-1., 0.2475},
{0., 0.25}, {1., 0.2475}, {2., 0.24}, {3., 0.2275}, {4., 0.21},
{5., 0.1875}, {6., 0.16}, {7., 0.1275}, {8., 0.09}, {9., 0.0475}, {10., 0.}}

Boundary condition: u(0,t)=u(1,t)=0.

```
uL[t_] := 0;
uR[t_] := 0;
```

```
uWA[-Nx, m_, alfa_] := uL[m * ht];
uWA[Nx, m_, alfa_] := uR[m * ht];
```

The numerical algorithm (see Eq.(19) of Ref[2])

```
RR[j_, m_, gamma_] := Module[{kk},
  kk = Sum[((1 - lambda) * w[k + 1, 1 - gamma] + lambda * w[k, 1 - gamma]) *
     (uWA[j - 1, m - 1 - k, gamma] - 2 * uWA[j, m - 1 - k, gamma] +
       uWA[j + 1, m - 1 - k, gamma]), {k, 0, m - 1}];
  uWA[j, m - 1, gamma] + sfrac * kk]
```

uWA[j, m, gamma] provides the value of the numeral solution at the point x_j, at time t_m for the case with anomalous diffusion exponent gamma

```
uWALista[m_, gamma_] :=
 uWALista[m, gamma] = LinearSolve[SparseArray[{Band[{1, 2}] → Table[
        Which[j == Nx - 1, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)], {j, -Nx, Nx - 1}],
      Band[{1, 1}] → Table[Which[j == -Nx, 1, j == Nx, 1, True,
          (1 + 2 * sfrac * w[0, 1 - gamma] * (1 - lambda))], {j, -Nx, Nx}],
      Band[{2, 1}] → Table[Which[j == -Nx, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)],
        {j, -Nx, Nx - 1}]}, 2 * Nx + 1], Table[
    Which[j == -Nx, uL[m * ht], j == Nx, uR[m * ht], True, RR[j, m, gamma]], {j, -Nx, Nx}]]
```

```
uWA[j_, m_, gamma_] := uWA[j, m, gamma] = uWALista[m, gamma][[Nx + 1 + j]]
```

```
uWALista[2, gamma]
```

```
{0.00622676, 0.0444768, 0.0855247, 0.12274, 0.155195, 0.182689, 0.205188,
 0.222688, 0.235188, 0.242688, 0.245188, 0.242688, 0.235188, 0.222688,
 0.205188, 0.182689, 0.155195, 0.12274, 0.0855247, 0.0444768, 0.00622676}
```

```
mfin = 1; soluWA[mfin * ht, gamma]
```

t=3.0625×10^{-6} : mfin=1 : ht=3.0625×10^{-6} : Delta x=0.05
 : Weight factor λ=0.8 : Computing time=0. : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.0446988}, {0.1, 0.0868886}, {0.15, 0.124354},
 {0.2, 0.15685}, {0.25, 0.18435}, {0.3, 0.20685}, {0.35, 0.22435},
 {0.4, 0.23685}, {0.45, 0.24435}, {0.5, 0.24685}, {0.55, 0.24435},
 {0.6, 0.23685}, {0.65, 0.22435}, {0.7, 0.20685}, {0.75, 0.18435},
 {0.8, 0.15685}, {0.85, 0.124354}, {0.9, 0.0868886}, {0.95, 0.0446988}, {1., 0}}
```

```
mfin = 10; soluWA[mfin * ht, gamma]
```

t=0.000030625 : mfin=10 : ht=3.0625×10^{-6} : Delta x=0.05
 : Weight factor λ=0.8 : Computing time=0.015625 : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.0417964}, {0.1, 0.0810504}, {0.15, 0.117237},
 {0.2, 0.148819}, {0.25, 0.176048}, {0.3, 0.198359}, {0.35, 0.215792},
 {0.4, 0.228263}, {0.45, 0.235751}, {0.5, 0.238248}, {0.55, 0.235751},
 {0.6, 0.228263}, {0.65, 0.215792}, {0.7, 0.198359}, {0.75, 0.176048},
 {0.8, 0.148819}, {0.85, 0.117237}, {0.9, 0.0810504}, {0.95, 0.0417964}, {1., 0}}
```

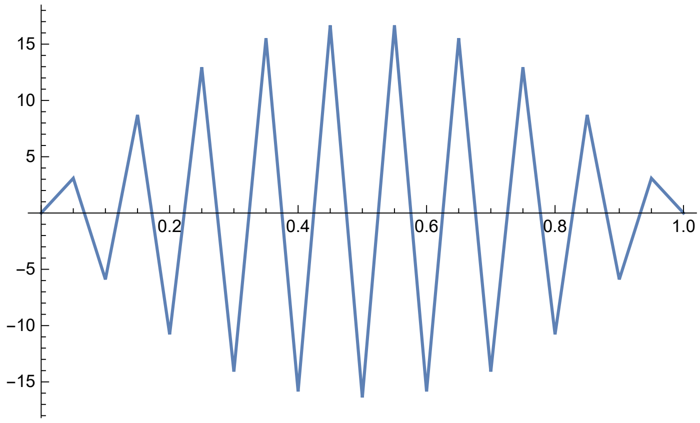
```
mfin = 100; soluWA[mfin * ht, gamma]
```

```
t=0.00030625 : mfin=100 : ht=3.0625×10^-6 : Delta x=0.05
  : Weight factor λ=0.8 : Computing time=0.640625 : Weighted-Averaged method
```

```
{{0., 0}, {0.05, 3.0896}, {0.1, -5.89788}, {0.15, 8.717},
 {0.2, -10.7797}, {0.25, 12.9501}, {0.3, -14.09}, {0.35, 15.5291},
 {0.4, -15.853}, {0.45, 16.6741}, {0.5, -16.3834}, {0.55, 16.6741},
 {0.6, -15.853}, {0.65, 15.5291}, {0.7, -14.09}, {0.75, 12.9501},
 {0.8, -10.7797}, {0.85, 8.717}, {0.9, -5.89788}, {0.95, 3.0896}, {1., 0}}
```

```
mfin = 100;
ListPlot[soluWA[mfin * ht, gamma], Joined → True, PlotRange → All]
```

```
t=0.00030625 : mfin=100 : ht=3.0625×10^-6 : Delta x=0.05
  : Weight factor λ=0.8 : Computing time=0. : Weighted-Averaged method
```



This bad result is due to the fact that the WA method is unstable for this case

Numerical solution of the FDE by means of the weighted-averaged method with weight factor λ=0.8 for

γ=1/2, K=1, , sfrac=0.55 , Δx=1/20 , Δt=1.89063×10^{-6}

The algorithm is stable in this case

Here we clear the saved values of uWALista and uWA (the evaluation of this cell is required if you are going to work with several examples in a Mathematica session)

```
Clear[uWALista, uWA];
uWALista[m_, gamma_] :=
 uWALista[m, gamma] = LinearSolve[SparseArray[{Band[{1, 2}] → Table[
        Which[j == Nx - 1, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)], {j, -Nx, Nx - 1}],
      Band[{1, 1}] → Table[Which[j == -Nx, 1, j == Nx, 1, True,
          (1 + 2 * sfrac * w[0, 1 - gamma] * (1 - lambda))], {j, -Nx, Nx}],
      Band[{2, 1}] → Table[Which[j == -Nx, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)],
        {j, -Nx, Nx - 1}]}, 2 * Nx + 1],
    Table[Which[j == -Nx, uL[m * ht], j == Nx, uR[m * ht], True, RR[j, m, gamma]],
     {j, -Nx, Nx}]]
uWA[j_, m_, gamma_] := uWA[j, m, gamma] = uWALista[m, gamma][[Nx + 1 + j]]
```

lambda=weight factor=λ

```
lambda = 0.8;
```

2*Nx+1=number of spatial points

```
Nx = 10;
```

Parameters of the example : gamma = 1/2, Kg = 1, sfrac = 0.55, deltax = 1./(2*Nx)

Notation: gamma=γ, ht=Δt , Kg=K , sfrac=S , deltax=Δx

```
{gamma = 0.5, deltax = 1. / (2 * Nx), Kg = 1.;
 sfrac = 0.55, ht = (sfrac * deltax^2 / Kg) ^ (1 / gamma)}
```

```
{0.5, 0.05, 0.55, 1.89063 × 10^-6}
```

According to Eq. (27) of Ref[2], the critical stability value for sfrac≡S=K(Δt)$^\gamma$/(Δx)^2 is given by

$S_\times$ = 1/(2*(2*lambda-1)* w(-1,1-gamma)

As we use the BDF1 coefficients, we have

w(-1,1-gamma)=2^(1-gamma)

The present numerical method is stable if 1 / sfracEfect ≥ 1 / $S_\times$ with sfracEfect = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)]) ^2 (see Eq. (25) of Ref[2])

```
{(Sin[(2 * Nx - 1) * Pi / (4 * Nx)]) ^2 // N;
 sfracEfectivo = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)]) ^2;
 scritico = 1 / (2 * (2 * lambda - 1) * 2^ (1 - gamma));
 1 / sfracEfectivo, 1 / scritico}
```

```
{1.82944, 1.69706}
```

Here we see that 1 / sfracEfect = 1.82944 is larger than 1 / $S_\times$ = 1.69706 . Therefore, the algorithm is stable

Initial condition : u(x,t=0)= x(1-x)

```
uWA[nx_, 0, alfa_] := (nx + Nx) * deltax * (1 - (nx + Nx) * deltax)
```

```
Table[{nx, uWA[nx, 0, 1]}, {nx, -Nx, Nx}] // N
```

```
{{-10., 0.}, {-9., 0.0475}, {-8., 0.09}, {-7., 0.1275}, {-6., 0.16},
 {-5., 0.1875}, {-4., 0.21}, {-3., 0.2275}, {-2., 0.24}, {-1., 0.2475},
 {0., 0.25}, {1., 0.2475}, {2., 0.24}, {3., 0.2275}, {4., 0.21},
 {5., 0.1875}, {6., 0.16}, {7., 0.1275}, {8., 0.09}, {9., 0.0475}, {10., 0.}}
```

Boundary condition: u(0,t)=u(1,t)=0.

```
uL[t_] := 0;
uR[t_] := 0;
```

```
uWA[-Nx, m_, alfa_] := uL[m * ht];
uWA[Nx, m_, alfa_] := uR[m * ht];
```

The numerical algorithm (see Eq.(19) of Ref[2])

```
RR[j_, m_, gamma_] := Module[{kk},
  kk = Sum[((1 - lambda) * w[k + 1, 1 - gamma] + lambda * w[k, 1 - gamma]) *
     (uWA[j - 1, m - 1 - k, gamma] - 2 * uWA[j, m - 1 - k, gamma] +
       uWA[j + 1, m - 1 - k, gamma]), {k, 0, m - 1}];
  uWA[j, m - 1, gamma] + sfrac * kk]
```

uWA[j, m, gamma] provides the value of the numeral solution at the point x_j, at time t_m for the case with anomalous diffusion exponent gamma

```
uWALista[m_, gamma_] :=
 uWALista[m, gamma] = LinearSolve[SparseArray[{Band[{1, 2}] → Table[
        Which[j == Nx - 1, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)], {j, -Nx, Nx - 1}],
      Band[{1, 1}] → Table[Which[j == -Nx, 1, j == Nx, 1, True,
         (1 + 2 * sfrac * w[0, 1 - gamma] * (1 - lambda))], {j, -Nx, Nx}],
      Band[{2, 1}] → Table[Which[j == -Nx, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)],
        {j, -Nx, Nx - 1}]}, 2 * Nx + 1], Table[
    Which[j == -Nx, uL[m * ht], j == Nx, uR[m * ht], True, RR[j, m, gamma]], {j, -Nx, Nx}]]
```

```
uWA[j_, m_, gamma_] := uWA[j, m, gamma] = uWALista[m, gamma][[Nx + 1 + j]]
```

```
uWALista[2, gamma]
```

```
{0.00493969, 0.0449063, 0.0864034, 0.123743, 0.156222, 0.183719, 0.206219,
 0.223719, 0.236219, 0.243719, 0.246219, 0.243719, 0.236219, 0.223719,
 0.206219, 0.183719, 0.156222, 0.123743, 0.0864034, 0.0449062, 0.00493969}
```

```
mfin = 1; soluWA[mfin * ht, gamma]
```

t=1.89063×10^-6 : mfin=1 : ht=1.89063×10^-6 : Delta x=0.05
 : Weight factor λ=0.8 : Computing time=0. : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.04525}, {0.1, 0.0875455}, {0.15, 0.125027},
 {0.2, 0.157525}, {0.25, 0.185025}, {0.3, 0.207525}, {0.35, 0.225025},
 {0.4, 0.237525}, {0.45, 0.245025}, {0.5, 0.247525}, {0.55, 0.245025},
 {0.6, 0.237525}, {0.65, 0.225025}, {0.7, 0.207525}, {0.75, 0.185025},
 {0.8, 0.157525}, {0.85, 0.125027}, {0.9, 0.0875455}, {0.95, 0.04525}, {1., 0}}
```

```
mfin = 10; soluWA[mfin * ht, gamma]
```

t=0.0000189063 : mfin=10 : ht=1.89063×10^-6 : Delta x=0.05
 : Weight factor λ=0.8 : Computing time=0.015625 : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.0425736}, {0.1, 0.0826737}, {0.15, 0.119096},
 {0.2, 0.151094}, {0.25, 0.178398}, {0.3, 0.200811}, {0.35, 0.21828},
 {0.4, 0.230768}, {0.45, 0.238264}, {0.5, 0.240763}, {0.55, 0.238264},
 {0.6, 0.230768}, {0.65, 0.21828}, {0.7, 0.200811}, {0.75, 0.178398},
 {0.8, 0.151094}, {0.85, 0.119096}, {0.9, 0.0826737}, {0.95, 0.0425736}, {1., 0}}
```

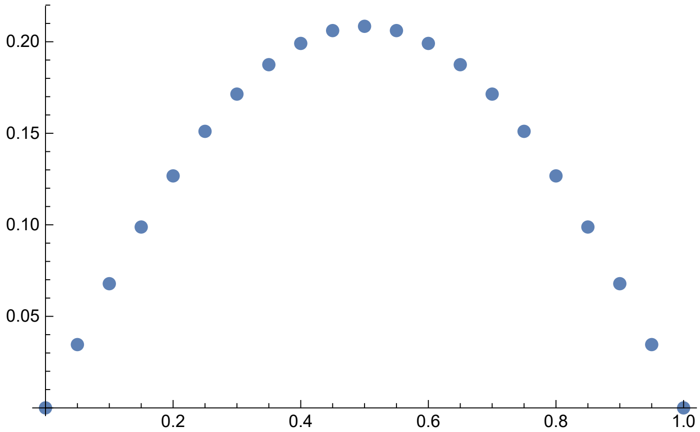
```
mfin = 100; soluWA[mfin * ht, gamma]
```

```
t=0.000189063 : mfin=100 : ht=1.89063×10^-6 : Delta x=0.05
  : Weight factor λ=0.8 : Computing time=0.578125 : Weighted-Averaged method
```

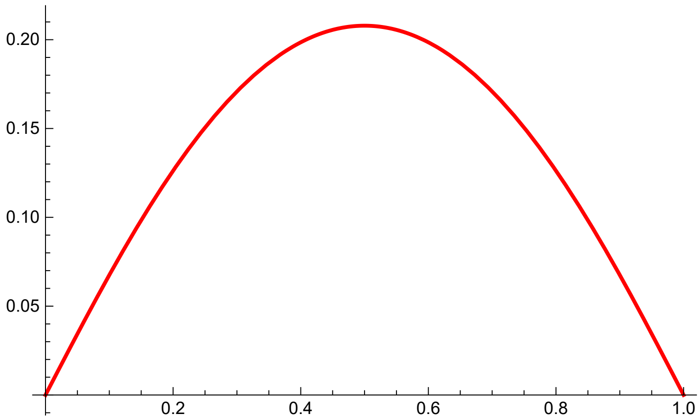
```
{{0., 0}, {0.05, 0.0370053}, {0.1, 0.0724716}, {0.15, 0.105321},
 {0.2, 0.134788}, {0.25, 0.160336}, {0.3, 0.1816}, {0.35, 0.198336},
 {0.4, 0.210385}, {0.45, 0.217649}, {0.5, 0.220075}, {0.55, 0.217649},
 {0.6, 0.210385}, {0.65, 0.198336}, {0.7, 0.1816}, {0.75, 0.160336},
 {0.8, 0.134788}, {0.85, 0.105321}, {0.9, 0.0724716}, {0.95, 0.0370053}, {1., 0}}
```

```
mfin = 200;
figNum = ListPlot[soluWA[mfin * ht, gamma], PlotRange → All, PlotStyle → PointSize[0.02]]
```

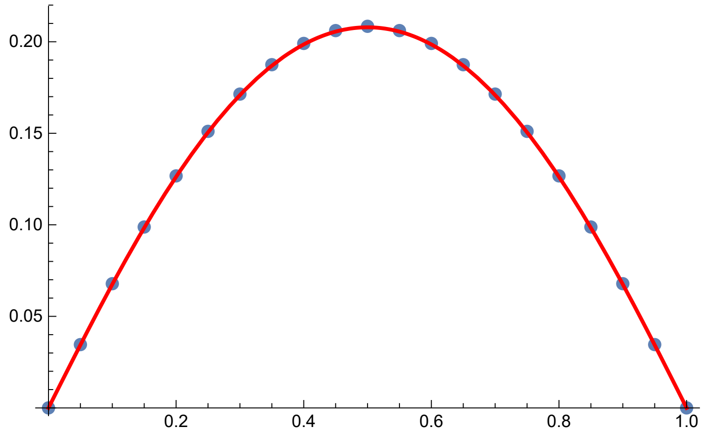
```
t=0.000378125 : mfin=200 : ht=1.89063×10^-6 : Delta x=0.05
  : Weight factor λ=0.8 : Computing time=1.70313 : Weighted-Averaged method
```



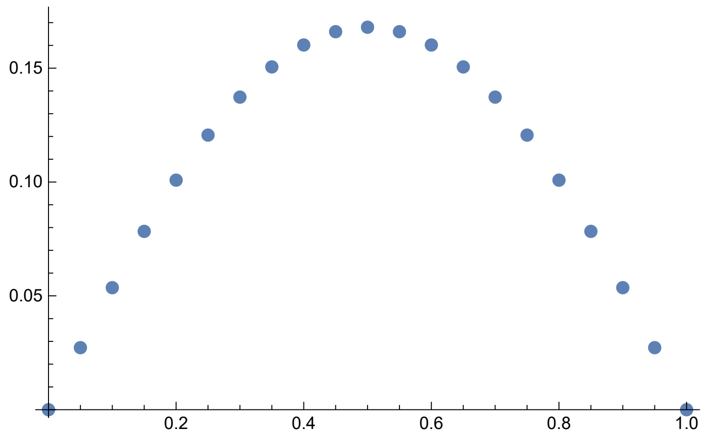

```
tv = 200 * ht;
figExact = Plot[uExactx1mx[x, tv, 1 / 2, 1, 20],
  {x, 0, 1}, PlotStyle → {Red, Thick}, PlotRange → All]
```



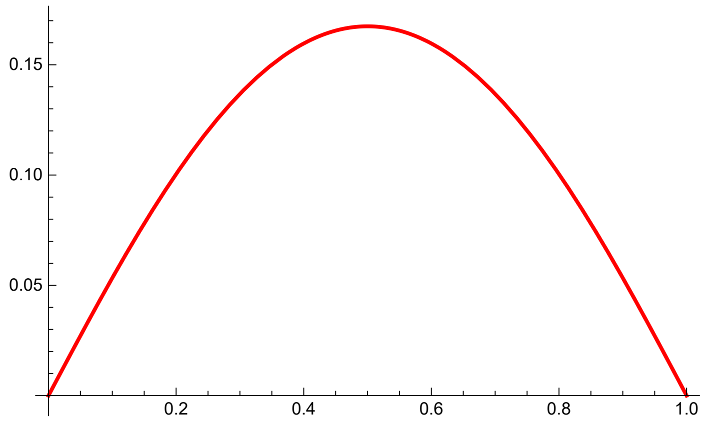
```
Show[figNum, figExact, PlotRange → All]
```



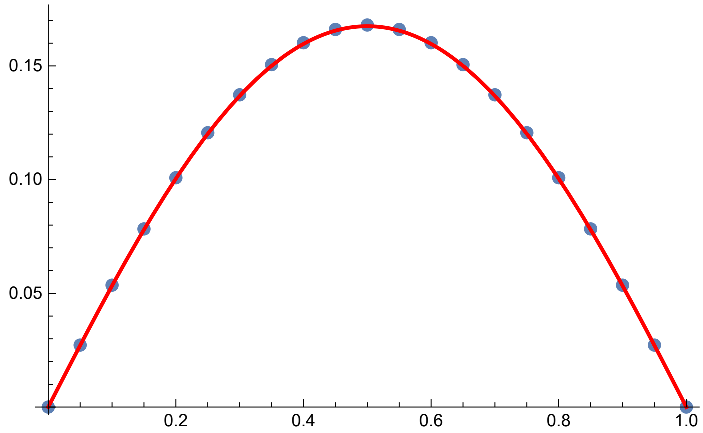

```
mfin = 500; soluWA[mfin * ht, gamma]

t=0.000945313 : mfin=500 : ht=1.89063×10^-6 : Delta x=0.05
  : Weight factor λ=0.8 : Computing time=12.7813 : Weighted-Averaged method

{{0., 0}, {0.05, 0.0306992}, {0.1, 0.0603336}, {0.15, 0.0880621},
 {0.2, 0.113213}, {0.25, 0.135256}, {0.3, 0.153782}, {0.35, 0.168483},
 {0.4, 0.179134}, {0.45, 0.185583}, {0.5, 0.187742}, {0.55, 0.185583},
 {0.6, 0.179134}, {0.65, 0.168483}, {0.7, 0.153782}, {0.75, 0.135256},
 {0.8, 0.113213}, {0.85, 0.0880621}, {0.9, 0.0603336}, {0.95, 0.0306992}, {1., 0}}

mfin = 1000; soluWA[mfin * ht, gamma]

t=0.00189063 : mfin=1000 : ht=1.89063×10^-6 : Delta x=0.05
  : Weight factor λ=0.8 : Computing time=43.2031 : Weighted-Averaged method

{{0., 0}, {0.05, 0.0272435}, {0.1, 0.0535941}, {0.15, 0.0783211},
 {0.2, 0.100822}, {0.25, 0.120608}, {0.3, 0.137286}, {0.35, 0.150555},
 {0.4, 0.160187}, {0.45, 0.166028}, {0.5, 0.167985}, {0.55, 0.166028},
 {0.6, 0.160187}, {0.65, 0.150555}, {0.7, 0.137286}, {0.75, 0.120608},
 {0.8, 0.100822}, {0.85, 0.0783211}, {0.9, 0.0535941}, {0.95, 0.0272435}, {1., 0}}

mfin = 1000;
figNum = ListPlot[soluWA[mfin * ht, gamma], PlotRange → All, PlotStyle → PointSize[0.02]]

t=0.00189063 : mfin=1000 : ht=1.89063×10^-6 : Delta x=0.05
  : Weight factor λ=0.8 : Computing time=0.03125 : Weighted-Averaged method
```



```
tv = 1000 * ht;
figExact = Plot[uExactx1mx[x, tv, 1 / 2, 1, 20],
  {x, 0, 1}, PlotStyle → {Red, Thick}, PlotRange → All]
```




```
Show[figNum, figExact, PlotRange → All]
```



Numerical solution of the FDE by means of the weighted-averaged method with weight factor λ=0, i.e., by means of a *fully* implicit method, for

γ=1/2, K=1, , sfrac=0.7 , Δx=1/20 , $\Delta t = 6.25 \times 10^{-6}$.

Weighted-averaged methods with $\lambda \leq 1/2$ are *always* stable; see section 3.1 of Ref[2]

Here we clear the saved values of uWALista and uWA (the evaluation of this cell is required if you are going to work with several examples in a Mathematica session)

```
Clear[uWALista, uWA];
uWALista[m_, gamma_] :=
 uWALista[m, gamma] = LinearSolve[SparseArray[{Band[{1, 2}] → Table[
       Which[j == Nx - 1, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)], {j, -Nx, Nx - 1}],
     Band[{1, 1}] → Table[Which[j == -Nx, 1, j == Nx, 1, True,
        (1 + 2 * sfrac * w[0, 1 - gamma] * (1 - lambda))], {j, -Nx, Nx}],
     Band[{2, 1}] → Table[Which[j == -Nx, 0, True, -sfrac * w[0, 1 - gamma] * (1 - lambda)],
       {j, -Nx, Nx - 1}]}, 2 * Nx + 1],
   Table[Which[j == -Nx, uL[m * ht], j == Nx, uR[m * ht], True, RR[j, m, gamma]],
    {j, -Nx, Nx}]]
uWA[j_, m_, gamma_] := uWA[j, m, gamma] = uWALista[m, gamma][[Nx + 1 + j]]
```

lambda=weight factor=λ

```
lambda = 0;
```

2*Nx+1=number of spatial points

```
Nx = 10;
```

Parameters of the example : gamma = 1/2, Kg = 1, sfrac = 1, deltax = 1./(2*Nx) -- -> always STABLE

Notation: gamma=γ, ht=Δt , Kg=κ , sfrac=s , deltax=Δx

```
{gamma = 0.5, deltax = 1. / (2 * Nx), Kg = 1.;
 sfrac = 1, ht = (sfrac * deltax^2 / Kg) ^ (1 / gamma)}
```

{0.5, 0.05, 1, 6.25×10^{-6}}

According to Eq. (27) of Ref[2], the critical stability value for sfrac≡S=K(Δt)$^{\gamma}$/(Δx)^2 is given by

$S_\times$ = 1/(2*(2*lambda-1)* w(-1,1-gamma)

As we use the BDF1 coefficients, we have

w(-1,1-gamma)=2^(1-gamma)

The present numerical method is stable if 1 / sfracEfect ≥ 1 / $S_\times$ with sfracEfect = sfrac * (Sin[(2 * Nx − 1) * Pi / (4 * Nx)]) ^2 (see Eq. (25) of Ref[2])

```
{(Sin[(2 * Nx - 1) * Pi / (4 * Nx)]) ^2 // N;
 sfracEfectivo = sfrac * (Sin[(2 * Nx - 1) * Pi / (4 * Nx)]) ^2;
 scritico = 1 / (2 * (2 * lambda - 1) * 2^ (1 - gamma));
 1 / sfracEfectivo, 1 / scritico}
```

{$\text{Sec}\left[\frac{\pi}{40}\right]^2$, −2.82843}

Here we see that 1 / sfracEfect = $\text{Sec}\left[\frac{\pi}{40}\right]^2$ is larger than 1 / $S_\times$ = −2.82843 . Therefore, the algorithm is stable

Initial condition : u(x,t=0)= x(1-x)

```
uWA[nx_, 0, alfa_] := (nx + Nx) * deltax * (1 - (nx + Nx) * deltax)
```

```
Table[{nx, uWA[nx, 0, 1]}, {nx, -Nx, Nx}] // N
```

{{−10., 0.}, {−9., 0.0475}, {−8., 0.09}, {−7., 0.1275}, {−6., 0.16}, {−5., 0.1875}, {−4., 0.21}, {−3., 0.2275}, {−2., 0.24}, {−1., 0.2475}, {0., 0.25}, {1., 0.2475}, {2., 0.24}, {3., 0.2275}, {4., 0.21}, {5., 0.1875}, {6., 0.16}, {7., 0.1275}, {8., 0.09}, {9., 0.0475}, {10., 0.}}

Boundary condition: u(0,t)=u(1,t)=0.

```
uL[t_] := 0;
uR[t_] := 0;
```

```
uWA[-Nx, m_, alfa_] := uL[m * ht];
uWA[Nx, m_, alfa_] := uR[m * ht];
```

```
uWALista[2, gamma]
```

```
{0.0450096, 0.0450096, 0.0864264, 0.123462, 0.155766, 0.183183, 0.205649,
 0.223135, 0.235629, 0.243127, 0.245626, 0.243127, 0.235629, 0.223135,
 0.205649, 0.183183, 0.155766, 0.123462, 0.0864264, 0.0450096, 0.0450096}
```

```
mfin = 1; soluWA[mfin * ht, gamma]
```

t=6.25×10^{-6} : mfin=1 : ht=6.25×10^{-6} : Delta x=0.05
: Weight factor λ=0 : Computing time=0. : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.0459549}, {0.1, 0.0878647}, {0.15, 0.125139},
 {0.2, 0.157553}, {0.25, 0.18502}, {0.3, 0.207508}, {0.35, 0.225003},
 {0.4, 0.237501}, {0.45, 0.245}, {0.5, 0.2475}, {0.55, 0.245}, {0.6, 0.237501},
 {0.65, 0.225003}, {0.7, 0.207508}, {0.75, 0.18502}, {0.8, 0.157553},
 {0.85, 0.125139}, {0.9, 0.0878647}, {0.95, 0.0459549}, {1., 0}}
```

```
mfin = 10; soluWA[mfin * ht, gamma]
```

t=0.0000625 : mfin=10 : ht=6.25×10^{-6} : Delta x=0.05
: Weight factor λ=0 : Computing time=0. : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.0416212}, {0.1, 0.0807336}, {0.15, 0.116317},
 {0.2, 0.147751}, {0.25, 0.174667}, {0.3, 0.196852}, {0.35, 0.214185},
 {0.4, 0.2266}, {0.45, 0.23406}, {0.5, 0.236548}, {0.55, 0.23406},
 {0.6, 0.2266}, {0.65, 0.214185}, {0.7, 0.196852}, {0.75, 0.174667},
 {0.8, 0.147751}, {0.85, 0.116317}, {0.9, 0.0807336}, {0.95, 0.0416212}, {1., 0}}
```

```
mfin = 100; soluWA[mfin * ht, gamma]
```

t=0.000625 : mfin=100 : ht=6.25×10^{-6} : Delta x=0.05
: Weight factor λ=0 : Computing time=0.59375 : Weighted-Averaged method

```
{{0., 0}, {0.05, 0.0335497}, {0.1, 0.0657116}, {0.15, 0.0956164},
 {0.2, 0.12259}, {0.25, 0.146118}, {0.3, 0.165812}, {0.35, 0.181388},
 {0.4, 0.192646}, {0.45, 0.199451}, {0.5, 0.201728}, {0.55, 0.199451},
 {0.6, 0.192646}, {0.65, 0.181388}, {0.7, 0.165812}, {0.75, 0.146118},
 {0.8, 0.12259}, {0.85, 0.0956164}, {0.9, 0.0657116}, {0.95, 0.0335497}, {1., 0}}
```

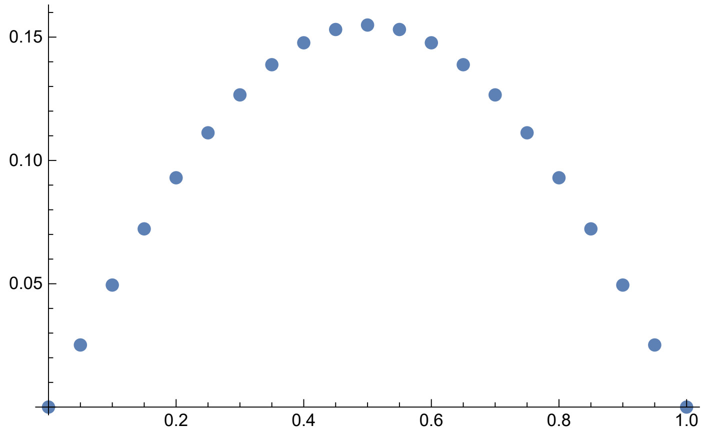
```
mfin = 500; soluWA[mfin * ht, gamma]
```

t=0.003125 : mfin=500 : ht=6.25×10^{-6} : Delta x=0.05
: Weight factor λ=0 : Computing time=13.25 : Weighted-Averaged method

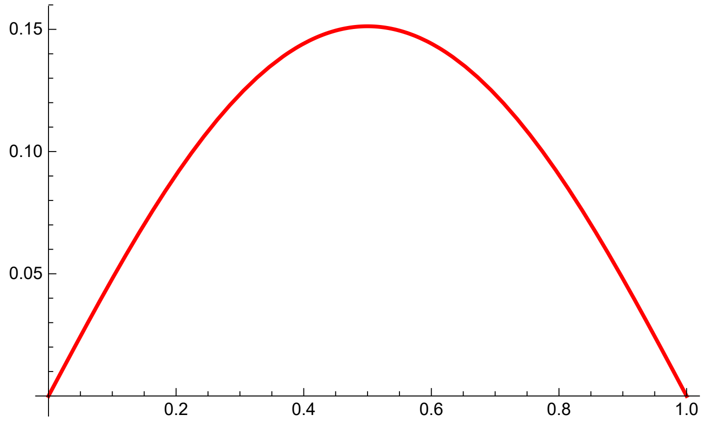
```
{{0., 0}, {0.05, 0.0251764}, {0.1, 0.0494732}, {0.15, 0.0722515},
 {0.2, 0.0929758}, {0.25, 0.111204}, {0.3, 0.126577}, {0.35, 0.138814},
 {0.4, 0.147703}, {0.45, 0.153094}, {0.5, 0.1549}, {0.55, 0.153094},
 {0.6, 0.147703}, {0.65, 0.138814}, {0.7, 0.126577}, {0.75, 0.111204},
 {0.8, 0.0929758}, {0.85, 0.0722515}, {0.9, 0.0494732}, {0.95, 0.0251764}, {1., 0}}
```

```
mfin = 500;
figNum = ListPlot[soluWA[mfin * ht, gamma], PlotRange → All, PlotStyle → PointSize[0.02]]
```

t=0.003125 : mfin=500 : ht=6.25×10^{-6} : Delta x=0.05
: Weight factor λ=0 : Computing time=0.015625 : Weighted-Averaged method




```
tv = 500 * ht;
figExact = Plot[uExactx1mx[x, tv, 1 / 2, 1, 20],
  {x, 0, 1}, PlotStyle → {Red, Thick}, PlotRange → All]
```




```
Show[figNum, figExact, PlotRange → All]
```