J. Phys. A: Math. Gen. **31** (1998) 6589–6593. Printed in the UK PII: S0305-4470(98)93819-7

# Demonstration of a conjecture for random walks in $d$-dimensional Sierpinski fractals

Josep M Porrà† and Santos B Yuste‡

† Departament de Física Fonamental, Universitat de Barcelona, Diagonal, 647, 08028-Barcelona, Spain

‡ Departamento de Física, Universidad de Extremadura, Avda. de Elvas, s/n, 06071-Badajoz, Spain

Received 30 April 1998

**Abstract.** Random walks on some fractals can be analysed by renormalization procedures. These techniques make it possible to obtain the Laplace transform of the first-passage time probability density function of a random walker that moves in the fractal. The calculation depends on a function $\rho(x)$ that is particular to each kind of fractal. For the Sierpinski family of fractals, it has been conjectured that $\rho(x) = 2dx^2 - 3(d-1)x + d - 2$, where $d$ is the dimension of the Euclidean space in which the Sierpinski fractal is embedded. We provide a proof of the conjecture that is based on the symmetries of the Sierpinski fractal.

The analysis of first-passage time probability density functions in fractals can be addressed by means of renormalization schemes [1–3]. These methods generally establish a relation between the distribution of times at different steps of the decimation of the fractal structure and lead to a function $\rho(x)$ that is characteristic of each type of fractal. This function relates the Laplace transform of the first-passage time probability density function (*pdf*) for the fractal decimated $k$ times, $\tilde{\psi}_k(s)$, and $k-1$ times, $\tilde{\psi}_{k-1}(s)$, in the following form $\tilde{\psi}_k(s) = 1/\rho[1/\tilde{\psi}_{k-1}(s)]$. The first-passage time *pdf* for the infinitely decimated fractal (i.e. when $k \to \infty$) can be obtained after solving the functional equation $1/\tilde{\psi}(\tau s) = \rho[1/\tilde{\psi}(s)]$. The factor $\tau$ gives the change in the timescale of the random walk after a decimation [3]. It has been conjectured [4] that, for the Sierpinski family of fractals, this function reads

$$\rho(x) = 2dx^2 - 3(d-1)x + d - 2 \tag{1}$$

where $d$ is the Euclidean dimension of the space in which the Sierpinski fractal is embedded. The conjecture has been checked for dimensions $d = 2$ [3], $d = 3$ [5], $d = 4$ and $d = 5$ [4]. We demonstrate the conjecture by an argument that rests on the symmetries of the Sierpinski fractal and the renormalization equations for the pausing time *pdf*.

We begin by defining the Sierpinski gasket in a Euclidean space of dimension $d$. The $d$-dimensional analogue of the triangle ($d = 2$) and the tetrahedron ($d = 3$) is the $d$-dimensional simplex [6]. It consists of $d+1$ vertices *all* connected to one another by segments that form its edges. For simplicity's sake, and without loss of generality, we will consider regular simplexes, that is, with all edges of the same length. The Sierpinski gasket in $d$ dimensions is constructed by replacing a simplex of edges of length $l$ by $d+1$ simplexes of edges length $l/2$. Each new simplex has a vertex in common with the original and $d$

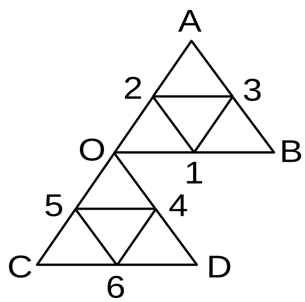


**Figure 1.** A part of an $n$-times decimated Sierpinski prefractal in two dimensions. The points identified by letters are $n$-vertices, i.e. vertices of the two-dimensional simplex (or triangle) of order $n$, and the points named by numbers disappear after the $n$th-order decimation. As discussed in the text, points 1, 2, 5 and 6 are of type $\alpha$ while 3 and 6 are of type $\beta$.

of its edges run along the edges of the primitive simplex. The new structure composed of $d + 1$ simplexes is called the generator. The mathematical self-similar Sierpinski fractal is achieved by iterating this process in both the inward and outward direction, that is, joining $d + 1$ of the iterated structures to build a new strucutre of characteristic size twice the original one [7]. Then, the fractal dimension $d_f$ of a $d$-dimensional Sierpinski fractal is $d_f = \log_2(d + 1)$. The object that arises after $n$ iterations of the generator is called a prefractal of order $n$ or equivalently a fractal with $n$ generations. The inverse process of passing from the fractal with $n$ generations to that with $n - 1$ is called decimation. In fact, renormalization methods are usually applied to the decimation procedure [3]. The notation of the decimation steps is the inverse of that of fractal generations. For example, the triangle OAB in figure 1 is the $n$th decimated fractal if the structure bounded by the triangles O12, A23 and 1B3 compose a Sierpinski prefractal decimated $n - 1$ times.

We consider a particle that performs a continuous time random walk on a Sierpinski lattice decimated $n$ times. After a randomly distributed time the particle jumps to any of its nearest neighbours with equal probability. We denote by $\psi_n(t)$ the *pdf* of the time between jumps of the random walker in the $n$th decimated Sierpinski fractal. The aim of the renormalization procedures is to relate $\psi_n(t)$ and $\psi_{n-1}(t)$. There are several schemes to achieve such a relation [1, 2, 5, 8, 9] but we exploit the method explained in [8]. The renormalization equations are written in the Laplace domain for time in order to deal with products instead of convolutions. The relation between the Laplace transform of $\psi_n(t)$, $\tilde{\psi}_n(s)$, and $\tilde{\psi}_{n-1}(s)$ originates the function $\rho(x)$ that is the aim of this paper. The argument $s$ of the Laplace transforms will be omitted for the sake of readability from now on.

The structure of the generator of a $d$-dimensional Sierpinski fractal is basic in order to write the renormalization equations. As was described above, the generator is built by adding to the $d+1$ vertices of the original simplex the intermediate points of all their edges. These new points are vertices of the $d + 1$ simplexes that constitute the generator. As the total number of initial points is $d+1$, there will be $d(d+1)/2$ new points. By construction, each new point of the generator has $2d$ nearest neighbours (*nn*). These new points can be classified as two types. The first type, $\alpha$, groups those points that are midpoints of the edges that end at the origin. As there are $d$ edges that begin at the origin, there are $d$ new points of type $\alpha$. The second type, $\beta$, is constituted by the midpoints of all edges of the original simplex that do not have the origin as a vertex. The total number of points of the second type is $d(d + 1)/2 - d = d(d - 1)/2$. Figure 1 shows two generators of the Sierpinski gasket in $d = 2$. The structure bounded by the triangle OAB is a generator. Points 1 and 2 are of type $\alpha$ while 3 is of type $\beta$.

We now proceed by identifying the type of *nn* of each new point of the generator. A point of type $\alpha$ has the origin, one vertex of the original simplex, $d-1$ points of type $\alpha$ and $d-1$ points of type $\beta$ as *nn*. Indeed, by definition of a point of type $\alpha$, the origin and one vertex of the original simplex are *nn* of it. Then, all other points of type $\alpha$ have the origin as common vertex with the point under consideration. Therefore, the $d-1$ other points of type $\alpha$ are *nn* by definition of simplex. The rest of *nn*, $d-1$, must be of type $\beta$ because no vertex of the original simplex besides the origin and the vertex of the edge considered are *nn*. The points of type $\beta$ have two vertices of the original simplex, two points of type $\alpha$ and $2d-4$ points of type $\beta$ as *nn*. Essentially, by definition of a point of type $\beta$, two vertices other than the origin are *nn* of the point considered. Each of these vertices has $d-2$ points of type $\beta$ as *nn*, excluding the one we are inspecting. Again, by the very nature of a simplex, these $2d-4$ points become *nn* of the point considered. Finally, as the two vertices of the edge that contains the point of type $\beta$ analysed have an edge that connects them with the origin, the midpoints of these edges, which are of type $\alpha$, must be *nn* of the point under analysis. These two points exhaust the identification of the *nn* of new points of type $\beta$.

Once the structure of a $d$-dimensional Sierpinski gasket has been elucidated, we continue to obtain the function $\rho(x)$. We consider a random walker that starts at the origin O in an $n$-times decimated Sierpinski fractal. In what follows, the vertices of the simplex that bounds the $n$th decimated fractal will be identified as $n$-vertices. The Laplace transform of the *pdf* of the first arrival time at any $n$-vertex is given by $\tilde{\psi}_n$. Let us call $\tilde{\psi}_\alpha$ the Laplace transform of the *pdf* of the first arrival time at any $n$-vertex from a point (an $(n-1)$-vertex) of type $\alpha$. The function $\tilde{\psi}_\beta$ has an identical definition but the random walker leaves from a point of type $\beta$. Figure 1 helps us to understand the different quantities involved. Let us assume that A, B, C, and D are $n$-vertices. The time it takes a random walker to reach any of these $n$-vertices from the origin O is distributed according to $\psi_n(t)$. If the random walker begins at a point of type $\alpha$ (points 1, 2, 5 and 6 of figure 1), the escape time is characterized by $\psi_\alpha(t)$. Finally, $\psi_\beta(t)$ gives the *pdf* of the time to reach any $n$-vertex from the points 3 and 4, the two points of type $\beta$ in figure 1. A probabilistic argument leads to the equations that link these probability density functions.

All *nn* of the origin O are of type $\alpha$. Consequently, the time taken to reach any $n$-vertex will be the addition of two independent times. First, the time to jump from the origin to a vertex of type $\alpha$ and, second, the time to go from a point of type $\alpha$ to any $n$-vertex (excluding the origin). These two times are distributed according to $\psi_{n-1}(t)$ and $\psi_\alpha(t)$, respectively. Then, the following equation holds:

$$\psi_n(t) = \int_0^t \psi_{n-1}(t')\psi_\alpha(t-t')\,\mathrm{d}t'. \tag{2}$$

The probability that the random walker reaches, between time $(t, t+\mathrm{d}t)$, any $n$-vertex from a point of type $\alpha$ is the convolution of the probability that it stays for a time $t'$ at this point, $(t' < t)$, jumps to a *nn* and takes a time $t - t'$ to reach any $n$-vertex from this point. The only exception to this reasoning arises when the random walker jumps, after a time $t'$, directly to a close $n$-vertex. In this case, $t' = t$. The analysis of the *nn* of a new point developed above and the fact that all *nn* are equally probable allows us to write an equation for $\psi_\alpha(t)$. This equation reads

$$\psi_\alpha(t) = \frac{1}{2d}\psi_{n-1}(t) + \int_0^t \left[\frac{1}{2d}\psi_{n-1}(t')\psi_n(t-t') + \frac{d-1}{2d}\psi_{n-1}(t')\psi_\alpha(t-t') + \frac{d-1}{2d}\psi_{n-1}(t')\psi_\beta(t-t')\right] \mathrm{d}t'. \tag{3}$$

A similiar analysis leads to the following equation for $\psi_\beta(t)$:

$$\psi_\beta(t) = \frac{1}{d}\psi_{n-1}(t) + \int_0^t \left[\frac{1}{d}\psi_{n-1}(t')\psi_\alpha(t-t') + \frac{d-2}{d}\psi_{n-1}(t')\psi_\beta(t-t')\right] \mathrm{d}t'. \tag{4}$$

The system of equations (2)–(4) is easily solved in the Laplace domain, where it has the following expression:

$$\begin{aligned}
\tilde{\psi}_n &= \tilde{\psi}_{n-1}\tilde{\psi}_\alpha \\
\tilde{\psi}_\alpha &= \frac{1}{2d}\tilde{\psi}_{n-1} + \frac{1}{2d}\tilde{\psi}_{n-1}\tilde{\psi}_n + \frac{d-1}{2d}\tilde{\psi}_{n-1}\tilde{\psi}_\alpha + \frac{d-1}{2d}\tilde{\psi}_{n-1}\tilde{\psi}_\beta \\
\tilde{\psi}_\beta &= \frac{1}{d}\tilde{\psi}_{n-1} + \frac{1}{d}\tilde{\psi}_{n-1}\tilde{\psi}_\alpha + \frac{d-2}{d}\tilde{\psi}_{n-1}\tilde{\psi}_\beta.
\end{aligned} \tag{5}$$

The expression of $1/\tilde{\psi}_n$ as a function $1/\tilde{\psi}_{n-1}$ is precisely equation (1) with $x = 1/\tilde{\psi}_{n-1}$. It is straightforward to show from the system (5) that

$$[2d^2x^3 + (5d - 3d^2)x^2 + (d^2 - 5d + 3)x + (d-2)]\tilde{\psi}_n = dx + 1 \tag{6}$$

and

$$\frac{1}{\tilde{\psi}_n} \equiv \rho(x) = 2dx^2 - 3(d-1)x + d - 2. \tag{7}$$

Therefore the conjecture (1) has been proved for any dimension $d$. This proof was the main aim of this paper.

This demonstration makes it possible to proceed further in the study of the properties of diffusion in Sierpinski fractals. For example, the average time $T_n$ to reach any $n$-vertex as a function of the average time $T_{n-1}$ that the particle stays at a point on the fractal decimated $n-1$ times can be obtained from equation (7). When $s \to 0$, the expansions $\tilde{\psi}_n \sim 1 - T_n s$ and $\tilde{\psi}_{n-1} \sim 1 - T_{n-1}s$ hold. Higher moments could be similarly obtained. Introducing these expansions into equation (7) and comparing the factors of order $s$, the relation $T_n = (d+3)T_{n-1}$ is established. This means that when the distance is scaled by a factor $\lambda = 2$, the average time for a random walker to cover such a distance increases by a factor $\tau = d + 3$. Therefore, the dimension of a random walk $d_w$ in a $d$-dimensional Sierpinski fractal is $d_w = \log\tau/\log\lambda = \log_2(d+3)$ [9].

**Acknowledgments**

JMP is grateful to Professor Josep Llosa for having introduced him to fractal geometry and for the careful reading of the manuscript. This work was partially supported by the Dirección General de Investigación Científica y Técnica (Spain) under grants PB96-0188 (JMP) and PB97-1501 (SBY).

## References


[1] Machta J 1981 *Phys. Rev.* B **24** 5260
[2] Van den Broeck C 1989 *Phys. Rev.* A **40** 7334
[3] Van den Broeck C 1989 *Phys. Rev. Lett.* **62** 1421
[4] Yuste S B 1995 *J. Phys. A: Math. Gen.* **28** 7027
[5] Parrondo J M R, Martinez H L, Kawai R and Lindenberg K 1990 *Phys. Rev.* A **42** 723
[6] 1960 *McGraw-Hill Encyclopedia of Science and Technology* vol 10 (New York: McGraw-Hill)
[7] Mandelbrot B B 1982 *The Fractal Geometry of Nature* (New York: Freeman)
[8] Hughes H D 1995 *Random Walks in Random Environments* vol I (Oxford: Clarendon) p 308
[9] Havlin S and ben-Avraham D 1987 *Adv. Phys.* **36** 695