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QUASI-PURE-CUBIC OSCILLATORS STUDIED USING A KRYLOV–BOGOLIUBOV METHOD

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A method of Krylov–Bogoliubov type which gives the approximate solution in terms of Jacobi elliptic functions is used to study the quasi-pure-cubic oscillators: $\ddot{x}+c_3x^3+\varepsilon f(x,\dot{x})=0$. Explicit approximate solutions with the perturbative terms x, x^2, $|\dot{x}|\dot{x}$ (quadratic damping) and sgn $(\dot{x})$ (Coulomb damping) are given and are compared with the numerical (or exact) solution. A simple and accurate expression is derived for the time at which the oscillations with Coulomb damping cease.

1. INTRODUCTION

Most analytical methods for the study of non-linear oscillators are applicable only to the quasi-linear (weakly non-linear) case

$$\ddot{x}+c_1x+\varepsilon f(x,\dot{x})=0, \tag{1.1}$$

where ε is a small parameter and $c_1>0$. This paper is concerned with a less well studied class of (strongly) non-linear oscillators, the quasi-pure-cubic oscillators, the equation for which is

$$\ddot{x}+c_3x^3+\varepsilon f(x,\dot{x})=0, \tag{1.2}$$

where ε is a small parameter and $c_3>0$. Some work on this equation has been published recently [1–4]. The method to be used is the elliptic Krylov–Bogoliubov (EKB) method [5], the formulae of which for approximate solutions will be summarized in section 2. It must be noted that in the method expounded in reference [5] the averaged equation of the phase is not correct when $c_1\neq 0$ *and* $c_3\neq 0$ because the averaging procedure is applied to an expression that is not periodic. Coppola and Rand have shown how to surmount this difficulty in reference [6]. In section 3 the solution and other interesting features (obtained by using the EKB method) of quasi-pure-cubic oscillator with four typical [7, 8] perturbative terms are presented. Quasi-pure-cubic oscillators with $f(x,\dot{x})=\dot{x}$ (linear damping) and $f(x,\dot{x})=(\alpha-\beta x^2)\dot{x}$ (van der Pol oscillator) have already been studied in reference [5].

2. THE EKB METHOD FOR QUASI-PURE-CUBIC OSCILLATORS

The EKB solution of equation (1.2) has the form [5]

$$x(t)=A(t)\ \mathrm{cn}\ [\psi(t),1/2]=A\ \mathrm{cn}, \tag{2.1}$$

where

$$\psi(t)=\Omega(t)-\phi(t),\qquad \Omega(t)=\int_0^t \omega(s)\,\mathrm{d}s,\qquad \omega^2=c_3A^2. \tag{2.2–2.4}$$

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The time derivative of the solution is

$$\dot{x} = -\omega A \text{ sn dn}. \tag{2.5}$$

The values of $A(t)$ and $\phi(t)$ are obtained by solving

$$\dot{A} = \frac{\varepsilon}{\omega}\frac{1}{4\mathrm{K}}\int_0^{4\mathrm{K}} f(A \text{ cn}, -A\omega \text{ sn dn}) \text{ sn dn d}\psi, \tag{2.6a}$$

$$\dot{\phi} = \frac{\varepsilon}{A\omega}\frac{1}{4\mathrm{K}}\int_0^{4\mathrm{K}} f(A \text{ cn}, -A\omega \text{ sn dn}) \text{ cn d}\psi, \tag{2.6b}$$

where $\mathrm{K}=\mathrm{K}(1/2)=1{\cdot}85407\ldots$ is the complete elliptic integral of the first kind of modulus $m=1/2$, and $\text{cn}=\text{cn}(\psi, 1/2)$, $\text{sn}=\text{sn}(\psi, 1/2)$ and $\text{dn}=\text{dn}(\psi, 1/2)$ are Jacobi elliptic functions of modulus $m=1/2$ [9, 10].

We consider two special types of quasi-pure-cubic oscillators (1.2), as follows.

(i) Type I: $f(x, \dot{x}) = f_1(x)$. From equations (2.6) one has

$$\dot{A}=0, \qquad \dot{\phi} = -\frac{\varepsilon}{A\omega}\frac{1}{4\mathrm{K}}\int_0^{4\mathrm{K}} f_1(A \text{ cn}) \text{ cn d}\psi. \tag{2.7a, b}$$

Defining

$$\Phi(A) = -\frac{1}{A\omega}\frac{1}{4\mathrm{K}}\int_0^{4\mathrm{K}} f_1(A \text{ cn}) \text{ cn d}\psi, \tag{2.8}$$

one finds that $\phi(t)=\varepsilon\Phi t+\phi_0$, and then the EKB approximate solution is

$$x(t) = A \text{ cn } [(\omega-\varepsilon\Phi)t-\phi_0, 1/2]. \tag{2.9}$$

(A zero subscript in a function denotes the value of this function at the initial time $t=0$.) The values of A and ϕ_0 are obtained from the initial conditions of the oscillation.

(ii) Type II: $f(x, \dot{x}) = f_2(\dot{x})$. For this case, equations (2.6) become

$$\dot{A} = \frac{\varepsilon}{\omega}\frac{1}{4\mathrm{K}}\int_0^{4\mathrm{K}} f_2(-A\omega \text{ sn dn}) \text{ sn dn d}\psi, \qquad \dot{\phi}=0. \tag{2.10a, b}$$

Then the solution is

$$x(t) = A(t) \text{ cn } [\Omega(t)-\phi_0, 1/2], \tag{2.11}$$

where

$$\Omega(t) = \int_0^t \omega(s)\,\mathrm{d}s = \sqrt{c_3}\int_0^t A(s)\,\mathrm{d}s \tag{2.12}$$

and $A(t)$ is the solution of equation (2.10a).

3. EXAMPLES

3.1. EXAMPLE A: $f(x, \dot{x}) = x$

The equation is

$$\ddot{x} + c_3x^3 + \varepsilon x = 0. \tag{3.1}$$

The oscillator is of type I. From equation (2.8) one finds that [10]

$$\Phi = -\frac{1}{\omega}\frac{1}{4K}\int_0^{4K} \mathrm{cn}^2\, d\psi = -\frac{2E-K}{\omega K} = -\frac{\alpha}{\omega}, \tag{3.2}$$

where $K = K(1/2)$ and $E = E(1/2) = 1{\cdot}35064\ldots$ is the complete elliptic integral of the second kind with modulus $m = 1/2$. Then $\alpha = 0{\cdot}4569\ldots$. The approximate EKB method solution is

$$x(t) = A\ \mathrm{cn}\ [(1 + \varepsilon\alpha/\omega^2)\omega t - \phi_0,\ 1/2]. \tag{3.3}$$

As is known, the exact solution of this oscillator is

$$x(t) = A\ \mathrm{cn}\ [\sigma t - \phi_0,\ m] \tag{3.4}$$

with $\sigma^2 = \varepsilon + c_3 A^2$ and $m = c_3 A^2/(2\sigma^2)$.

In Figures 1 and 2 are plotted the approximate solution given by equation (3.3) and the exact solution given by equation (3.4). The approximate solution is better for larger oscillations because the perturbative term is less important. The differences between the approximate and exact solutions come principally from the difference between the periods

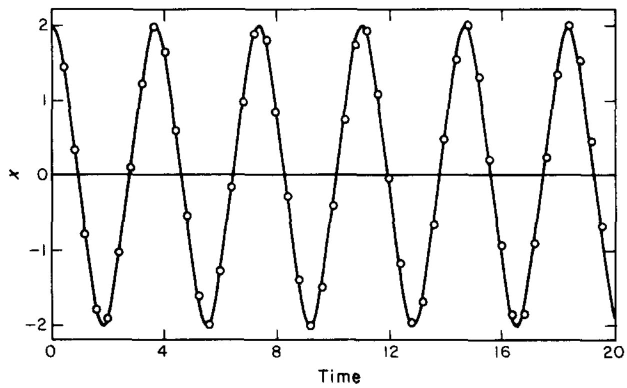

Figure 1. Approximate (——) and exact (O) solutions of the quasi-cubic-oscillator $\ddot{x} + x^3 + 0{\cdot}1x = 0$, with initial conditions $x(0) = 2$ and $\dot{x}(0) = 0$. The approximate solution is obtained by using formula (3.3). The exact solution is obtained by using formula (3.4).

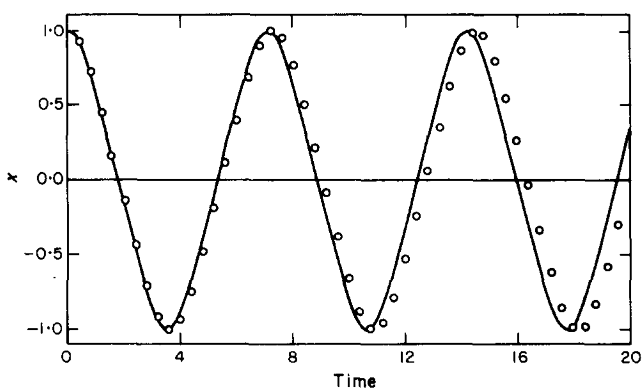

Figure 2. Approximate (——) and exact (O) solutions of the quasi-cubic-oscillator $\ddot{x} + x^3 + 0{\cdot}1x = 0$, with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. These solution are obtained as indicated in the caption to Figure 1.

of the two solutions. In the oscillator of Figure 1 the approximate period, $T_a = 4K(1/2)/[\omega + (\varepsilon\alpha)/\omega]$ is $T_a = 3{\cdot}62$, whereas the exact $T = 4K(m)/\sigma$ is $T = 3{\cdot}64$. However, for the oscillator of Figure 2, the difference is, as expected, larger: $T = 6{\cdot}9$ and $T_a = 7{\cdot}1$.

3.2. EXAMPLE B: $f(x, \dot{x}) = x^2$

The equation is

$$\ddot{x} + c_3 x^3 + \varepsilon x^2 = 0. \tag{3.5}$$

The oscillator is of type I. From equation (2.8) one easily finds [10] that $\Phi = 0$. Then the solution is given by

$$x(t) = A \text{ cn } [\omega t - \phi_0, 1/2] \tag{3.6}$$

with $\omega^2 = c_3 A^2$. This is the *exact* solution of the non-perturbed ($\varepsilon = 0$) oscillator. A similar result is obtained with this same perturbative term x^2 for quasi-linear oscillators: the Krylov–Bogoliubov method gives an approximate solution that is the exact solution of the linear (non-perturbed, $\varepsilon = 0$) oscillator [7]. In Figures 3 and 4 are plotted the approximate solution given by equation (3.6) and the numerical solution obtained by using a fourth order Runge–Kutta method. As in example A, the approximate solution is worse for

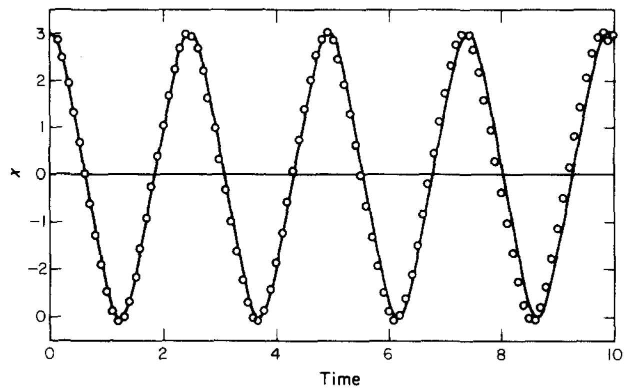

Figure 3. Approximate (——) and numerical (O) solutions of the quasi-cubic-oscillator $\ddot{x} + x^3 + 0{\cdot}1x^2 = 0$, with initial conditions $x(0) = 3$ and $\dot{x}(0) = 0$. The approximate solution is obtained by using formula (3.6). The numerical solution is obtained by using a Runge–Kutta method of fourth order.

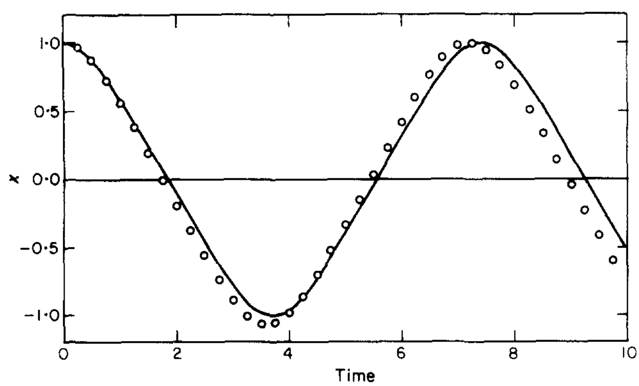

Figure 4. Approximate (——) and numerical (O) solutions of the quasi-cubic-oscillator $\ddot{x} + x^3 + 0{\cdot}1x^2 = 0$, with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. These solution are obtained as indicated in the caption to Figure 3.

smaller amplitudes because the perturbation εx^2 of the cubic force is comparatively larger. For quasi-linear oscillators the opposite is the case: smaller amplitudes imply better approximate solutions.

3.3. EXAMPLE C: $f(x, \dot{x}) = |\dot{x}|\dot{x}$ (QUADRATIC DAMPING)

The oscillator is

$$\ddot{x} + c_3 x^3 + \varepsilon |\dot{x}| \dot{x} = 0. \tag{3.7}$$

The oscillator is of type II. The amplitude equation (2.10a) is [10]

$$\dot{A} = -\varepsilon A^2 \omega \frac{1}{4K} \int_0^{4K} |\mathrm{sn}\,\mathrm{dn}|\ \mathrm{sn}^2\,\mathrm{dn}^2\,\mathrm{d}\psi = -\tilde{\varepsilon} A^3, \tag{3.8}$$

where

$$\tilde{\varepsilon} = (2\varepsilon/5K)\sqrt{c_3}. \tag{3.9}$$

Integrating expression (3.8) one obtains

$$A(t) = A_0/[1 + 2\tilde{\varepsilon} A_0^2 t]^{1/2}, \tag{3.10}$$

with

$$\Omega(t) = (\sqrt{c_3}/\tilde{\varepsilon})[(1/A) - (1/A_0)]. \tag{3.11}$$

It is of interest to note that the decaying amplitude for the quasi-pure-cubic oscillator is proportional to $t^{-1/2}$, whereas for the quasi-linear oscillator it is proportional to t^{-1} [7, 8]. In Figures 5–7 are plotted the fourth order Runge–Kutta numerical solution and the approximate solution given by equation (2.11) with equations (3.10) and (3.11). The approximate solutions are good even for perturbative parameters as large as $\varepsilon = 1$.

3.4. EXAMPLE D: $f(x, \dot{x}) = \mathrm{sgn}(\dot{x})$ (COULOMB DAMPING)

The equation is

$$\ddot{x} + c_3 x^3 + \varepsilon\ \mathrm{sgn}(\dot{x}) = 0. \tag{3.12}$$

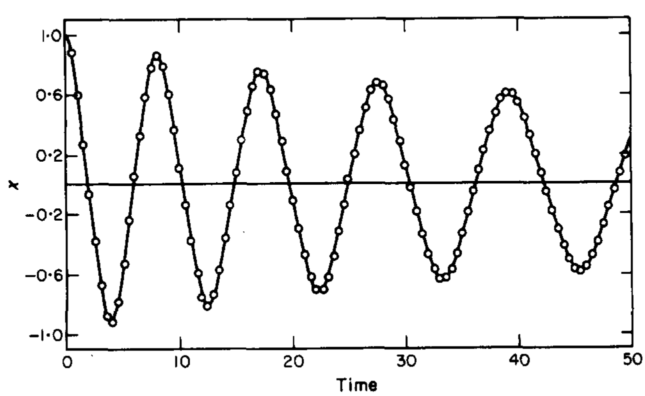

Figure 5. Approximate (——) and numerical (○) solutions of the quasi-cubic-oscillator $\ddot{x} + x^3 + 0{\cdot}1|\dot{x}|\dot{x}$, with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. The approximate solution is obtained by using formula (2.11) with expressions (3.10) and (3.11). The numerical solution is obtained by using a Runge–Kutta method of fourth order.

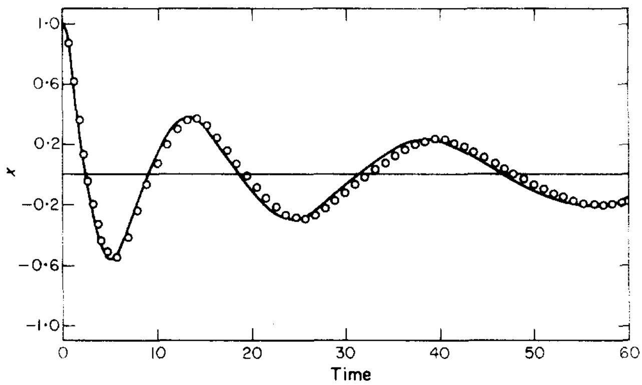

Figure 6. Approximate (——) and numerical (○) solutions of the quasi-cubic-oscillator $\ddot{x}+x^3+|\dot{x}|\dot{x}=0$, with initial conditions $x(0)=1$ and $\dot{x}(0)=0$. These solution are obtained as indicated in the caption to Figure 5.

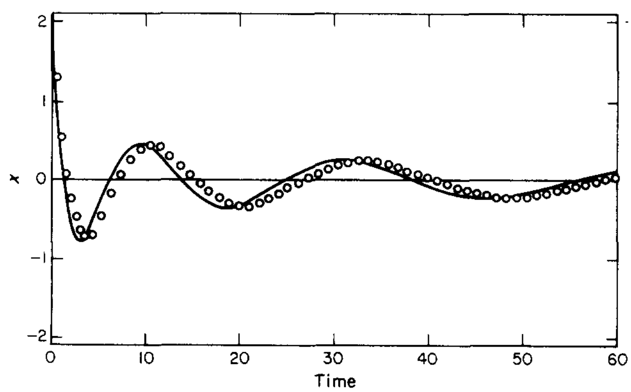

Figure 7. Approximate (——) and numerical (○) solutions of the quasi-cubic-oscillator $\ddot{x}+x^3+|\dot{x}|\dot{x}=0$, with initial conditions $x(0)=2$ and $\dot{x}(0)=0$. These solution are obtained as indicated in the caption to Figure 5.

The oscillator is of type II. The amplitude equation is [10]

$$\dot{A}=\frac{\varepsilon}{\omega}\frac{1}{4\mathrm{K}}\int_0^{4\mathrm{K}}\operatorname{sgn}\,(-A\omega\,\mathrm{sn}\,\mathrm{dn})\,\mathrm{sn}\,\mathrm{dn}\,\mathrm{d}\psi=-\frac{\tilde{\varepsilon}}{A},\tag{3.13}$$

where

$$\tilde{\varepsilon}=\varepsilon/(\sqrt{c_3}\,\mathrm{K}).\tag{3.14}$$

Integrating, one finds

$$A=\sqrt{A_0^2-2\tilde{\varepsilon}t}\cdot\tag{3.15}$$

The oscillations cease at the time t_d where $A=0$; that is,

$$t_d=A_0^2/2\tilde{\varepsilon}.\tag{3.16}$$

Substituting equation (3.15) in equation (2.12) one obtains

$$\Omega(t)=(\sqrt{c_3}/3\tilde{\varepsilon})(A_0^3-A^3).\tag{3.17}$$

TABLE 1

Detention time for the oscillator $\ddot{x}+x^3+\varepsilon\,\mathrm{sgn}\,(\dot{x})=0$ evaluated by numerical integration, $[t_d]_n$, and by means of the approximate expression (3.16), $[t_d]_a=KA_0^2/(2\varepsilon)$, for several values of the initial amplitude A_0 and perturbative parameter ε

ε	A_0	$[t_d]_n$	$[t_d]_n$
0.1	4	148.4	148.3
0.1	2	37.7	37.1
0.2	2	18.4	18.5
0.3	2	12.5	12.4
0.5	4	30.0	29.7

The approximate solution is then

$$x(t)=\begin{cases} A(t)\ \mathrm{cn}\ [\Omega(t)-\phi_0,\ 1/2], & 0\leqslant t\leqslant t_d \\ 0, & t\geqslant t_d \end{cases}, \tag{3.18}$$

where A and Ω given by expressions (3.15) and (3.17).

In Table 1 are listed the detention times evaluated numerically, $[t_d]_n$, and approximately by expression (3.16), $[t_d]_a$, for different oscillators and initial conditions. The detention time is proportional to A_0^2, in contrast with the quasi-linear oscillator for which it is proportional to A_0 [7, 8]. In Figures 8–11 are plotted the approximate solutions given by expression (3.18) and the numerical solutions obtained by using a fourth order Runge–Kutta method. The approximate solutions are good for a wide range of perturbative parameters ε and initial amplitudes.

4. CONCLUSIONS

The EKB method applied to the quasi-pure-cubic oscillators is fruitful (remember, for example, the accurate expression for the oscillation detention time of the Coulomb damped oscillator) and simple (we have been able to find analytic approximate solutions for the four oscillators studied). However there were two factors that made this method somewhat

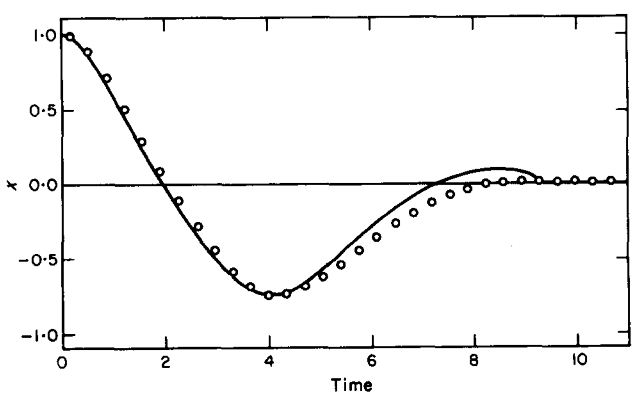

Figure 8. Approximate (——) and numerical (O) solutions of the quasi-cubic-oscillator $\ddot{x}+x^3+0{\cdot}1\ \mathrm{sgn}\ (\dot{x})=0$, with initial conditions $x(0)=1$ and $\dot{x}(0)=0$. The approximate solution is obtained by using formula (3.18). The numerical solution is obtained by using a Runge–Kutta method of fourth order.

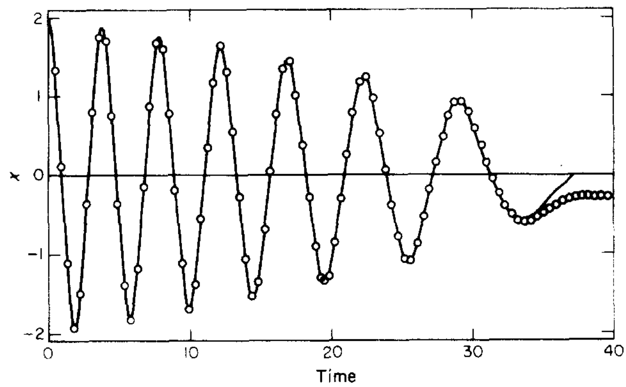

Figure 9. Approximate (——) and numerical (O) solutions of the quasi-cubic-oscillator $\ddot{x}+x^3+0{\cdot}1\ \mathrm{sgn}\ (\dot{x})=0$, with initial conditions $x(0)=2$ and $\dot{x}(0)=0$. These solution are obtained as indicated in the caption to Figure 8.

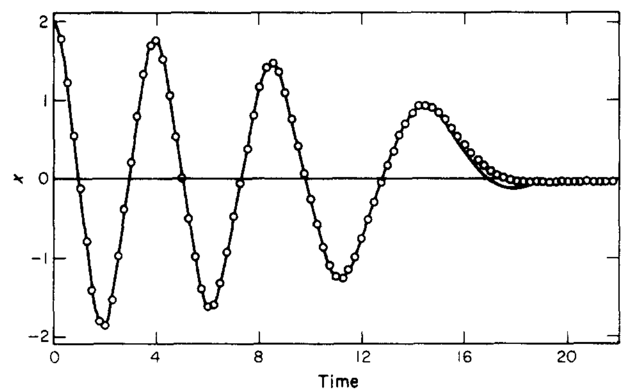

Figure 10. Approximate (——) and numerical (O) solutions of the quasi-cubic-oscillator $\ddot{x}+x^3+0{\cdot}2\ \mathrm{sgn}\ (\dot{x})=0$, with initial conditions $x(0)=2$ and $\dot{x}(0)=0$. These solution are obtained as indicated in the caption to Figure 8.

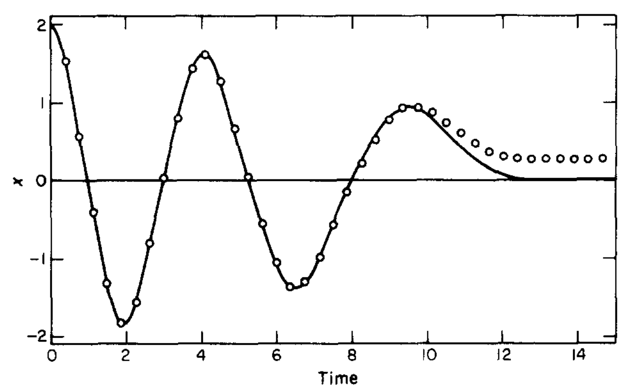

Figure 11. Approximate (——) and numerical (O) solutions of the quasi-cubic-oscillator $\ddot{x}+x^3+0{\cdot}3\ \mathrm{sgn}\ (\dot{x})=0$, with initial conditions $x(0)=2$ and $\dot{x}(0)=0$. These solution are obtained as indicated in the caption to Figure 8.

more difficult to use for quasi-pure-cubic oscillators than the usual KB method for quasi-linear oscillators. The first was that, in deriving the EKB approximate solution, we have to handle integrals with elliptic functions rather than circular functions (as in the usual KB method). The second was that, for quasi-pure-cubic oscillators where $\dot{A} \neq 0$, i.e., where the oscillation frequency is time-dependent, the explicit expression for the cn argument of the EKB approximate solution requires an extra integration, $\psi = \Omega - \phi$ with $\Omega = \int \omega(t)\, \mathrm{d}t$, as compared with the usual Krylov–Bogoliubov method in which $\Omega = \sqrt{c_1}\, t$. However, these difficulties are not very serious, as we have demonstrated in the examples of section 3.

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