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"CUBICATION" OF NON-LINEAR OSCILLATORS USING THE PRINCIPLE OF HARMONIC BALANCE

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Abstract—A new method is given of "cubication" of autonomous non-linear oscillators (NLO) of the class $\ddot{x} + c_1 x + c_3 x^3 + \varepsilon[g(x) + f(x)\dot{x}] = 0$, i.e. of constructing the cubic oscillator $\ddot{x} + \lambda^* \dot{x} + c_1^* x + c_3^* x^3 = 0$ from the NLO. The solution, limit cycles, bifurcations, fixed points, and stability of this NLO are approached by studying its associated cubic oscillator which is equal at least in the largest harmonics (principle of harmonic balance) and by assuming as a first approximation a solution for the NLO problem in terms of Jacobian elliptic functions. When $c_3 = 0$, the elliptic functions become circular functions and the present method reduces to the well-studied harmonic-balance method of linearization. The present method is equivalent to a third-order Chebyshev expansion of the NLO force if this is conservative. For a dissipative NLO, it gives the position and features of limit cycles and bifurcations.

1. INTRODUCTION

Most of the approximate quantitative methods used for solving the non-linear oscillator (NLO)

$$\ddot{x} + F(x, \dot{x}) = 0 \tag{1}$$

are applicable only if this oscillator is of the class

$$\ddot{x} + c_1 x + \varepsilon h(x, \dot{x}) = 0 \tag{2}$$

with $c_1 > 0$ and ε a small parameter. Some of these methods are termed linearization methods, because they obtain the linear oscillator

$$\ddot{x} + c_1^* x + \lambda^* \dot{x} = 0 \tag{3}$$

from equation (2) with suitable coefficients, and then the solution of this equation is taken as an approximation to that of equation (2). For example, one method of this kind is the Krylov–Bogoliubov method in the first appproximation [1, Chapter XII]. Another method [1–3] uses the principle of harmonic balance for obtaining equation (3) from equation (2); it is this method we shall call the linearization method.

Although the class of oscillators (2) is very useful, it is also limited as many important NLOs do not belong to it. The equation of the pendulum, for oscillations that are not very small, is a good example. A wider class of NLO is

$$\ddot{x} + c_1 x + c_3 x^3 + \varepsilon h(x, \dot{x}) = 0. \tag{4}$$

which, when $c_3 = O(\varepsilon)$, is equation (2).

In previous publications the author and his colleagues have presented some techniques for solving equation (4): a Krylov–Bogoliubov method that uses Jacobian elliptic functions as generating solutions [4], a Krylov–Bogoliubov method and a harmonic-balance method when $c_1 = 0$ [5], another harmonic-balance method when $h(x, \dot{x}) = f(x)\dot{x}$ [6], and a cubication technique when $h(x, \dot{x}) = g(x)$ [7]. This paper looks at a method in which the solution of equation (4) with $h(x, \dot{x}) = g(x) + f(x)\dot{x}$ (where $g(x)$ and $f(x)$ are analytic functions in the oscillation interval), i.e. the solution of the class of oscillators

$$\ddot{x} + c_1 x + c_3 x^3 + \varepsilon[g(x) + f(x)\dot{x}] = 0 \tag{5}$$

is approximated by the solution of the cubic oscillator

$$\ddot{x} + c_1^* x + c_3^* x^3 + \lambda^* \dot{x} = 0 \tag{6}$$

where c_1^*, c_3^*, and λ^* are obtained using the principle of harmonic balance. This is the cubication method of the present work. Comparison will be made with numerical integration and with the results given by other authors.

The linearization method that uses the principle of harmonic balance is known to be suitable for ascertaining the limit cycles and topological configuration of the solutions of NLOs of type (2). We will show that the cubication method proposed here also has these desirable properties for NLOs of type (5).

2. CUBICATION USING THE PRINCIPLE OF HARMONIC BALANCE

One seeks a cubic oscillator

$$\ddot{x} + F^*(x, \dot{x}) = 0$$

with

$$F^*(x, \dot{x}) = c_1^* x + c_3^* x^3 + \lambda^* \dot{x}$$

so that its solution is a good approximation to the solution of equation (1) with

$$F(x, \dot{x}) = c_1 x + c_3 x^3 + \varepsilon[g(x) + f(x)\dot{x}]. \tag{7}$$

If in equation (5) we set $\varepsilon = 0$, the resulting equation (generating equation) has the solution (generating solution) in terms of Jacobian elliptic functions,

$$x(t) = A\,\mathrm{cn}(\omega t + \theta, \mu^2) = A \cos z \tag{8}$$

where

$$z = \mathrm{am}(\omega t + \theta, \mu^2) \equiv \mathrm{am}(\psi, \mu^2)$$

and

$$\omega^2 = c_1 + c_3 A^2$$
$$\mu^2 = (c_3 A^2/2)/(c_1 + c_3 A^2).$$

Assuming that the solution of equation (5) is given as a first approximation by equation (8), the parameters c_1^*, c_3^*, and λ^* of equation (6) are obtained by equating

$$F(x, \dot{x}) = F(A\cos z, A\dot{\cos} z) \equiv F$$

and

$$F^*(x, \dot{x}) = F^*(A\cos z, A\dot{\cos} z) \equiv F^*$$

at least in their largest harmonics (principle of harmonic balance). The term $A\dot{\cos} z$ is equal to

$$A\omega[\mathrm{d}\,\mathrm{cn}(\psi, \mu^2)/\mathrm{d}\psi] = -A\omega\,\mathrm{sn}(\psi, \mu^2)\,\mathrm{dn}(\psi, \mu^2) = -\omega A \sin z\, \Delta z$$

with $\mathrm{sn}(\psi, \mu^2) = \sin z$ and $\mathrm{dn}(\psi, \mu^2) = \Delta z$. Writing h.o.h. for higher-order harmonics, as

$$F = a_1 \cos z + a_3 \cos 3z + b_1 \sin z + \text{h.o.h.} \tag{9}$$

$$F^* = \alpha_1 \cos z + \alpha_3 \cos 3z + \beta_1 \sin z + \text{h.o.h.} \tag{10}$$

with

$$\alpha_1 = c_1^* A + \tfrac{3}{4} c_3^* A^3$$
$$\alpha_3 = \tfrac{1}{4} c_3^* A^3$$
$$\beta_1 = -\lambda^* \omega A \gamma_1$$

where

$$\gamma_1 = (1/\pi)\int_0^{2\pi} \Delta z \sin^2 z\, \mathrm{d}z$$

and equating the coefficients of the largest harmonics in equations (9) and (10), one obtains

$$c_1^* = (a_1 - 3a_3)/A \tag{11}$$

$$c_3^* = 4a_3/A^3 \tag{12}$$

$$\lambda^* = -b_1/(\omega A \gamma_1). \tag{13}$$

From these last three equations, one sees that four definite integrals must be calculated to obtain c_1^*, c_3^*, and λ^*:

$$a_1 = (1/\pi)\int_0^{2\pi} F(A\cos z,\ -\omega A\sin z\,\Delta z)\cos z\,dz = \frac{1}{\pi}\int_0^{2\pi} F\cos z\,dz$$

$$a_3 = \frac{1}{\pi}\int_0^{2\pi} F\cos 3z\,dz$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} F\sin z\,dz \tag{14}$$

$$\gamma_1 = \frac{1}{\pi}\int_0^{2\pi} \sin^2 z\,\Delta z\,dz. \tag{15}$$

Substituting the force of equation (7) into equation (14) one has

$$b_1 = -(\varepsilon\omega A/\pi)\int_0^{2\pi} f(A\cos z,\ -\omega A\sin z\,\Delta z)\sin^2 z\,\Delta z\,dz. \tag{16}$$

Notice that in equations (9) and (10), although the coefficients of $\cos z$ and $\cos 3z$ are not negligible in general, the other harmonics are of $O(\varepsilon)$.

3. APPLICATION OF THE METHOD

3.1. *Conservative oscillators*

For the conservative NLO,

$$\ddot{x} + F(x) = 0 \tag{17}$$

one obtains, using equations (11) and (12), the associated oscillator

$$\ddot{x} + F^*(x) = 0$$

with

$$F^*(x) = c_1^* x + c_3^* x^3.$$

The solution of this last equation is the present method's approximate solution to equation (17). It is given by equation (8) with

$$\omega^2 = c_1^* + c_3^* A^2 \tag{18}$$

$$\mu^2 = (c_3^* A^2/2)/(c_1^* + c_3^* A^2). \tag{19}$$

It is known that the expression $F^*(x) = c_1^* x$ obtained by the linearization method is equal to the Chebyshev polynomial expansion of $F(x)$ in $[-A, A]$ truncated after the linear term. Also, as is easy to prove, the expression $F^*(x) = c_1^* x + c_3^* x^3$ obtained by the present method of cubication is equal to the Chebyshev polynomial expansion of $F(x)$ in $[-A, A]$ truncated after the cubic term. Therefore, we will not extend the discussion about this class of oscillators.

3.2. *Dissipative oscillators: limit cycles*

In Section 2 the principle of harmonic balance was used to obtain the cubic oscillator (6) from the oscillator under study, equation (5). The transient of this oscillator is then approximated by the transient of the cubic oscillator. But transients will not be studied here: we will focus on where harmonic balance is especially useful, in the search and study of limit cycles (also called closed orbits).

From equation (6) one sees that there is a limit cycle when $\lambda^* = 0$, i.e. by equation (16), when $b_1 = 0$, given by

$$x(t) = A_s \operatorname{cn}(\omega t + \theta, \mu^2) = A_s \operatorname{cn}(\psi, \mu^2) = A_s \cos z$$

$$\dot{x}(t) = -\omega A_s \operatorname{sn}(\psi, \mu^2)\operatorname{dn}(\psi, \mu^2) = -\omega A_s \sin z\,\Delta z \tag{20}$$

where A_s is the amplitude and ω and μ^2 are given by equations (18) and (19).

Let us now define some useful quantities: the non-linearity factor $\nu = c_3^* A^2/c_1^*$, the total energy of oscillation $En = c_1^* A^2 + c_3^* A^4/2 = c_1^*(1 + \nu/2)$, maximum or minimum potential $V_m = (c_1^*)^2/(2c_3^*)$, and the position x_m of this maximum or minimum, $x_m^2 = -c_1^*/c_3^*$.

In terms of the non-linearity factor ν, the frequency and parameter of equations (18) and (19) are

$$\omega^2 = c_1^*(1 + \nu) \tag{21}$$

$$\mu^2 = \nu/[2(1 + \nu)]. \tag{22}$$

According to the value of the parameter μ^2, the limit cycle (20) can be expressed in terms of Jacobian elliptic functions in three ways:

(i) $0 \le \mu^2 < 1$. This is the case when $0 \le \nu < \infty$, i.e. when $c_1^* > 0$, $c_3^* > 0$ (therefore, $0 \le \mu^2 < 1/2$) and when $\nu \le -2$, i.e. when $c_1^* < 0$, $c_3^* > 0$ and $En \ge 0$ (therefore, $1/2 \le \mu^2 \le 1$). The limit cycle is given by

$$x = A_s \operatorname{cn}(\psi, \mu^2) \tag{23}$$

$$\dot{x} = -A_s \omega \operatorname{sn}(\psi, \mu^2) \operatorname{dn}(\psi, \mu^2). \tag{24}$$

An expression that will be needed below is

$$dz = \operatorname{dn}(\psi, \mu^2)\, d\psi.$$

(ii) $\mu^2 \le 0$. This is the case when $-1 \le \nu \le 0$, i.e. when $c_1^* > 0, c_3^* < 0$ and $En < V_m$. We distinguish this case since it is not usual (although it would be possible) to work with elliptic function parameters outside the interval [0, 1]. For example, tables of elliptic functions are given only within this interval. When $\mu^2 \le 0$ we use the negative parameter transformation [8], and thus

$$\cos z = \operatorname{cd}(\psi_\sigma, \sigma^2) \equiv \operatorname{cd}$$

$$\sin z = \sigma_1 \operatorname{sd}(\psi_\sigma, \sigma^2) \equiv \sigma_1 \operatorname{sd}$$

$$\Delta z = \operatorname{nd}(\psi_\sigma, \sigma^2) \equiv \operatorname{nd}$$

$$dz = \sigma_1 \operatorname{nd}\, d\psi_\sigma \tag{25}$$

where $\sigma_1^2 = 1 - \sigma^2$, $\psi_\sigma = \psi/\sigma_1$ and $\sigma^2 = -\mu^2/(1 - \mu^2)$ or

$$\sigma^2 = -\nu/(2 + \nu). \tag{26}$$

Of course, now the parameter σ^2 of the elliptic functions satisfies $0 \le \sigma^2 \le 1$. From equations (20) the limit cycle is then given by

$$x = A_s \operatorname{cd} \tag{27}$$

$$\dot{x} = -\sigma_1 A_s \omega \operatorname{sd} \operatorname{nd}. \tag{28}$$

(iii) $\mu^2 \ge 1$. This is the case when $c_1^* < 0$, $c_3^* > 0$ and $En \le 0$, i.e. when $-2 \le \nu \le -1$. To have the elliptic function parameter inside the interval [0, 1], we use the reciprocal parameter transformation [8] and thus

$$\cos z = \operatorname{dn}(\psi_\eta, \eta^2) \equiv \operatorname{dn}$$

$$\sin z = \eta \operatorname{sn}(\psi_\eta, \eta^2) \equiv \eta \operatorname{sn}$$

$$\Delta z = \operatorname{cn}(\psi_\eta, \eta^2) \equiv \operatorname{cn}$$

$$dz = \eta \operatorname{cn}\, d\psi_\eta \tag{29}$$

where $\psi_\eta = \mu\psi$ and $\eta^2 = 1/\mu^2$ or

$$\eta^2 = 2(1 + \nu)/\nu \tag{30}$$

and now the parameter η^2 of the elliptic functions lies between zero and one. Then from equations (20) the limit cycle is given by

$$x = A_s \operatorname{dn} \tag{31}$$

$$\dot{x} = -\eta A_s \omega \operatorname{sn} \operatorname{cn}. \tag{32}$$

4. THE VAN DER POL–DUFFING OSCILLATOR

The first example is the van der Pol–Duffing oscillator

$$\ddot{x} + c_1 x + c_3 x^3 = \varepsilon(\alpha - \beta x^2)\dot{x}. \tag{33}$$

This interesting equation is a simple example of a differential equation with local and global bifurcations of codimension greater than one. Moreover, this NLO appears in the study of some problems of practical interest, such as flow-induced oscillations [9] or convection in diffusive systems [10].

A detailed analysis of this equation can be found in refs [11, 12], which used differential dynamics, and also in ref. [10] which uses the principle of energy balance.

From equations (11) and (12) one gets $c_1^* = c_1$ and $c_3^* = c_3$. From equations (33) and (16)

$$b_1 = -(\varepsilon\omega A/\pi)\int_0^{2\pi}(\alpha - \beta A^2\cos^2 z)\sin^2 z\,\Delta z\,\mathrm{d}z = -(\varepsilon\omega A/\pi)(\alpha Q - \beta T A^2)$$

and from equation (15)

$$\gamma_1 = Q/\pi$$

where

$$Q = \int_0^{2\pi}\sin^2 z\,\Delta z\,\mathrm{d}z$$

and

$$T = \int_0^{2\pi}\sin^2 z\cos^2 z\,\Delta z\,\mathrm{d}z.$$

Then from equation (13)

$$\lambda^* = \varepsilon[\alpha - \beta(T/Q)A^2]. \tag{34}$$

When one evaluates this expression three possibilities appear:

(i) The oscillator is hard ($c_1 > 0$, $c_3 > 0$) or soft–hard ($c_1 < 0$, $c_3 > 0$). Then $0 < \mu^2 < 1$ ($\nu > 0$ or $\nu < -2$) and (e.g. ref. [13])

$$Q = \int_0^{4K}\mathrm{sd}^2\,\mathrm{dn}^2\,\mathrm{d}\psi = (4/3\mu^2)[(2\mu^2 - 1)E + \mu_1^2 K]$$

$$T = \int_0^{4K}\mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{dn}^2\,\mathrm{d}\psi = (4/15\mu^4)[2(\mu^4 + \mu_1^2)E + \mu_1^2(\mu^2 - 2)K]$$

where $\mu_1^2 = 1 - \mu^2$, $K = K(\mu^2)$ is the complete elliptic integral of the first kind, and $E = E(\mu^2)$ is the complete elliptic integral of the second kind. Integration is over the period of $x(t)$, which, from equation (8), is $4K(\mu^2)$.

(ii) The oscillator is soft ($c_1 > 0$, $c_3 > 0$, $En < V_{\max}$). Therefore, $\mu^2 < 0\,(-1 < \nu < 0)$ and (e.g. ref. [13])

$$Q = \int_0^{4K}\sigma_1^3\,\mathrm{sn}^2\,\mathrm{nd}^2\,\mathrm{d}\psi_\sigma = (4/3\sigma^2\sigma_1)[(1 + \sigma^2)E - \sigma_1^2 K]$$

$$T = \int_0^{4K}\sigma_1^3\,\mathrm{sd}^2\,\mathrm{cd}^2\,\mathrm{nd}^2\,\mathrm{d}\psi_\sigma = (4/15\sigma^4\sigma_1)[2(\sigma^4 + \sigma_1^2)E + \sigma_1^2(\sigma^2 - 2)K]$$

where $K = K(\sigma^2)$ and $E = E(\sigma^2)$. The integration is over the period of $x(t)$, which, from equation (25), is $4K(\sigma^2)$.

(iii) The oscillator is soft–hard ($c_1 < 0$, $c_3 > 0$ with $En < 0$). Therefore, $\mu^2 > 1$ ($-2 < \nu < -1$) and (e.g. ref. [13])

$$Q = \int_0^{2K}\eta^3\,\mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{d}\psi_\eta = (2/3\eta)[(2 - \eta^2)E - 2\eta_1^2 K]$$

$$T = \int_0^{2K}\eta^3\,\mathrm{dn}^2\,\mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{d}\psi_\eta = (2/15\eta)[2(\eta^4 + \eta_1^2)E + \eta_1^2(\eta^2 - 2)K]$$

where $\eta_1^2 = 1 - \eta^2$, $K = K(\eta^2)$ and $E = E(\eta^2)$. The integration is over the period of $x(t)$, which, from equation (29), is $2K(\eta^2)$.

In short, the parameters c_1^*, c_3^* and λ^* for each kind (hard, soft and soft–hard) of van der Pol–Duffing oscillator (33) and for any energy, have been obtained as long as the motion is oscillatory.

The limit cycles are found by solving $\lambda^* = 0$. This equation, by equations (34), (21) and (18), is equivalent to the system

$$A^2 = (\alpha/\beta)[Q(v)/T(v)] \tag{35a}$$

$$A^2 = (c_1^*/c_3^*)v \equiv rv. \tag{35b}$$

Each solution (A_s^2, v_s) defines a limit cycle with its frequency ω given by equation (21), and its parameter, depending on the value of v_s, given by equation (22), (26), or (30). An important and useful feature of the system (35) is that the dissipative and conservative terms, equations (35a) and (35b), are considered separately.

In equations (35) there are two independent parameters: α/β and r. This method of cubication is well illustrated by studying how the limit cycles are affected by the variation of r for fixed α/β. In the following α/β is unity.

4.1. *Discussion of results*

Figure 1 shows the graphical solution of system (35). We have plotted what we call the "dissipative" curve or curve D (35a) and some "conservative" curves or curves C (35b). The curve D represents all the limit cycles that the dissipative force can generate in cubic oscillators; the point (A_s^2, v_s) where curve C cuts curve D defines the cycle actually generated.

A careful examination of Fig. 1 is instructive. First, one notices that curve D tends asymptotically to

$$A_\infty^2 = (5/3)\{2[E(1/2)/K(1/2)] - 1\} = 3.6474\ldots \tag{36}$$

when $v \to \pm\infty$, since then $\mu^2 \to 1/2$ and equation (35a) becomes equation (36). When the oscillator is purely cubic, $c_1 = c_1^* = 0$, $r = 0$ and curve C cuts curve D at $v = \pm\infty$. Therefore, the limit cycle is given by equations (23) and (24) with $A_s^2 = A_\infty^2$, $\mu^2 = 1/2$, and $\omega^2 = c_3 A_\infty^2$. This result was obtained in ref. [1]. For the more linear hard oscillators ($c_1 > 0, c_3 > 0$), r increases, A_s^2 increases, and v_s decreases. So, when $r = 1$, the limit cycle is given by equations (23) and (24) with $v_s = A_s^2 = 3.745$ and, therefore, with $\mu^2 = 0.395$ and $\omega^2 = 4.745c_1^*$. For the linear oscillator $c_3 = 0$, $r = \infty$, and then $v_s = 0$, $A_s^2 = 4$, $\mu^2 = 0$ and $\omega^2 = c_1^*$. It is not surprising that this should be the well-known result given by the

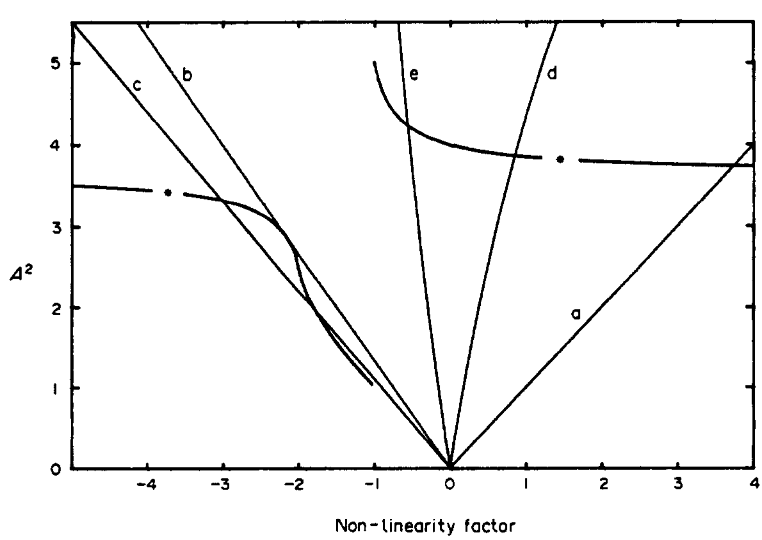

Fig. 1. Graphical solution of system (35): (— ∗ —) D-curve for a dissipative force of van der Pol type with $\alpha/\beta = 1$; C-curve for the conservative force $c_1x + c_3x^3$ with $r = 1$ labelled "a"; the same with $r = r_2 = -1.329$ labelled "b" the same with $r = -11/10$ labelled "c"; C-curve for the conservative force sinh x labelled "d"; C-curve for the conservative force sin x labelled "e".

linearization method because $\mathrm{cn}(\psi, \mu^2 = 0) = \cos\psi$. In other words, the linearization method using harmonic balance is a particular case of the present method.

For soft oscillators ($c_1 > 0$, $c_3 < 0$) there is periodic motion, i.e. a limit cycle, if $-1 < \nu \leq 0$, but not if $\nu \leq -1$. Also it is well known that $\pm x_m$ ($x_m^2 = -r$) are saddle points. As can be seen in Fig. 1, $5 \geq A_s^2 \geq 4$ when $-1 \leq \nu \leq 0$, and therefore the limit cycle given by equations (27) and (28) does not reach the saddle points if $-\infty < r < -5$. When $r = -5$, the limit cycle becomes a separatrix, i.e. a homoclinic cycle. For $r \geq -5$ there are no periodic solutions.

For soft–hard oscillators ($c_1 < 0, c_3 > 0, \nu \leq -1$) there are three singular points: a saddle point at $x = 0$ and two foci at $\pm x_m$. For brevity we will always say "foci", but these singular points are really nodes if $\varepsilon^2(\alpha + \beta c_1/c_3)^2 + 8c_1 > 0$. One observes in Fig. 1 that curve C with $r < r_2 = -1.329$ does not cut curve D in the region with $\nu < -1$, i.e. soft–hard oscillators with $r < r_2$ have no limit cycles. When $r = r_2$, curve C touches curve D at a point—coalescence point—with coordinates $A_s^2 = 2.877$, $\nu_s \equiv \nu_c = -2.164$. Therefore, the limit cycle given by equations (23) and (24) with the above amplitude, with $\omega^2 = -1.164c_1^*$ and $\mu^2 = 0.930$, is a semistable limit cycle. When $r = r_4 \equiv -5/4$ is reached, there is a cycle with $\nu_s < -2$ and another at $\nu_s = -2$ with $A_s^2 = 5/2$, $\omega^2 = -c_1$ and $\mu^2 = \eta^2 = 1$. This last is a separatrix (a homoclinic orbit: double-saddle loops). In the following, we use the term internals (externals) for limit cycles with $\nu_s > -2$ ($\nu_s < -2$). For $r > r_4$ there is one external cycle, given by equations (23) and (24), with $\nu_s < -2$ and $A_s^2 > 5/2$, and two internal ones, given by equations (31) and (32), with $-2 < \nu_s < -1$ and $1 < A_s^2 < 5/2$. For the right internal cycle $A_s = +(A_s^2)^{1/2}$, and for the left internal cycle $A_s = -(A_s^2)^{1/2}$. For example, for $c_1 = -11$, $c_3 = 10$, $\alpha = \beta = 1$, one has $r = -11/10 > r_4$ and there is one external cycle at $\nu_s = -3.019$, $A_s^2 = 3.321$ with $\omega^2 = 22.213$ and $\mu^2 = 0.748$, and two internal cycles at $\nu_s = -1.781$, $A_s^2 = 1.959$ with $\omega^2 = 8.890$, $\eta^2 = 0.877$ and $A_s = +(1.959)^{1/2}$ for one and $A_s = -(1.959)^{1/2}$ for the other. Figure 2 shows the above limit cycles and the limit cycles calculated numerically for $\varepsilon = 0.5$. Even for such a large ε the present method gives good results. Notice that the linearization method is not even applicable. One observes in Fig. 1 that, if r approaches $r_1 = -1$, then A_s^2 and ν_s approach one, i.e. the internal limit cycles contract onto the foci. When $r = r_1$, the internal cycles disappear into the foci that then change their stability character. For $r > -1$, only the external cycle remains. This is known in differential dynamics as Hopf bifurcation.

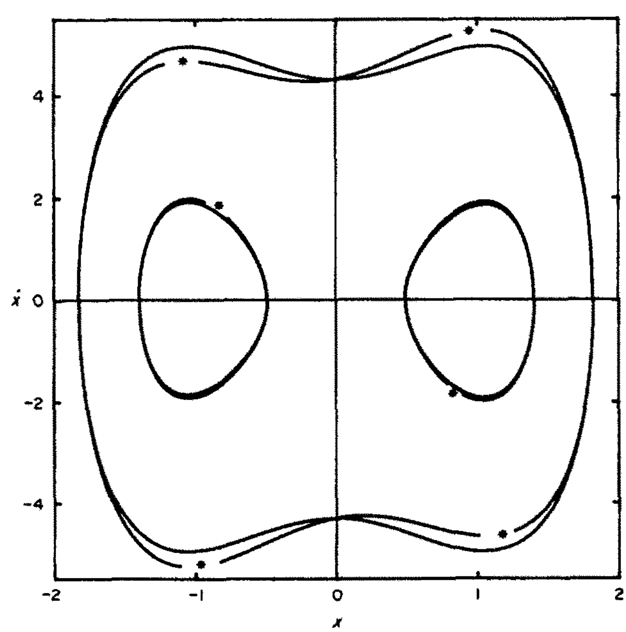

Fig. 2. Numerical and approximate analytical limit cycles for the example oscillator (33) with $c_1 = -11$, $c_3 = 10$, $\alpha = \beta = 1$ and $\varepsilon = 0.5$: (—•—), numerical limit cycles; (——), analytical limit cycles.

We have already seen how the parameter r affects the limit cycle configurations for a fixed $\alpha/\beta = 1$. We will now give some results for $\alpha/\beta > 0$. From equation (35a), for a given ν, $\alpha/\beta \neq 1$ only has the effect of stretching the curve D in Fig. 1 by the factor α/β. But the homoclinic cycle, the coalescence cycle, the double-saddle loops, and the Hopf bifurcation occur for $\nu = -1$, $\nu = 2.164$, $\nu = -2$, and $\nu = -1$, respectively, for any value of α/β. Then it is easy to deduce that the homoclinic cycle occurs when $\alpha/\beta = -r/5$, the coalescence when $\alpha/\beta = -(2.164/2.877)r$, the double-saddle loops when $\alpha/\beta = -4r/5$, and the Hopf bifurcation when $\alpha/\beta = -r$. These results coincide with those obtained using the methods of differential dynamics [14].

Finally let us consider the stability of the limit cycles and the singular points. For each oscillator, the function $\lambda^*(A)$ is given by equation (34). If at $\lambda^* = 0$ and $A = A_s$ $d\lambda^*/dA$ happens to be positive, the limit cycle is unstable (U), and if $d\lambda^*/dA$ is negative it is stable (S). In Fig. 3, we plot λ^* vs A for the oscillator with $c_1 = -11$, $c_3 = 10$, $\alpha = \beta = 1$ and $\varepsilon = 1$. Internal cycles are seen to be U and the external cycle is S (the foci are S). In general if $\varepsilon > 0$ and $c_1^* < 0$, the limit cycles with $\nu < \nu_c$ are S, with $\nu_c < \nu < -1$ are U, the foci at $\nu = -1$ are S if $r < -1$ and U if $r > -1$, and $(x, \dot{x}) = (0, 0)$ is a saddle point. For $\varepsilon > 0$ and $c_1^* > 0$, the limit cycle is S for all $\nu > -1$ and the focus at $(x, \dot{x}) = (0, 0)$ is U. Finally, if $\varepsilon < 0$, the sign of λ^* changes, and therefore the stability of all the above limit cycles and foci are reversed, i.e. U → S and S → U.

5. TWO OTHER EXAMPLES

The examples are:

$$\ddot{x} + \sinh x = \varepsilon(\alpha - \beta x^2)\dot{x} \tag{37}$$

$$\ddot{x} + \sin x = \varepsilon(\alpha - \beta x^2)\dot{x}. \tag{38}$$

As before we will set, without loss of generality, $\alpha = \beta = 1$. For oscillator (37), by equations (11) and (12), one gets [15]

$$c_1^*(A) = 2[I_1(A) - 3I_3(A)]/A$$

$$c_3^*(A) = 8I_3(A)/A^3$$

where $I_n(A) = (-i)^n J_n(iA)$ is a modified Bessel function. Using equation (13), one finds that $\lambda^*(A)$ is given by equation (34) again, since the dissipative force is the same in all the examples. That is, the dissipative curve given by equation (35a) is the same D-curve of the van der Pol–Duffing example of Section 4. The C-curve given by equation (35b) is plotted in Fig. 1. There is only one intersection: $A_s^2 = 3.867$ and $\nu_s = 0.854$, defining the limit cycle given by equations (23) and (24) with $\omega^2 = 1.772$ and $\mu^2 = 0.230$ since $c_1^*(A_s) = 0.956$.

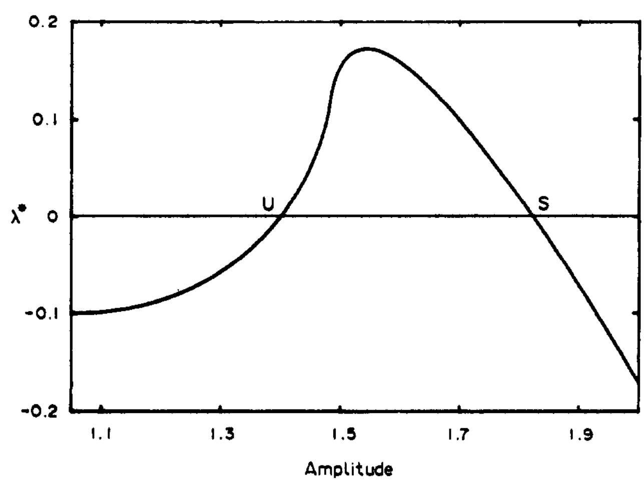

Fig. 3. A plot of λ^* vs amplitude for the example oscillator (33) with $c_1 = -11$, $c_3 = 10$, $\alpha = \beta = 1$ and $\varepsilon = 1$. The U-limit cycles are unstable and the S-limit cycle is stable.

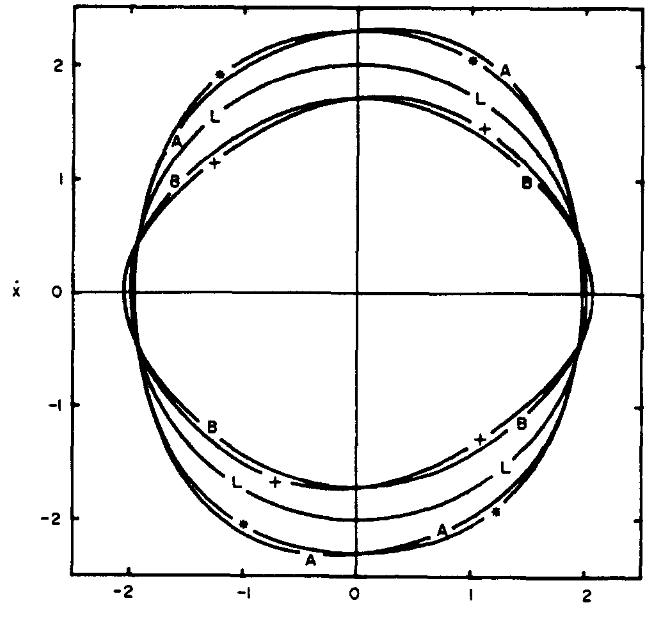

Fig. 4. Numerical and approximate analytical limit cycles for the example oscillators (37) and (38) with $\alpha = \beta = 1$ and $\varepsilon = 0.1$: (— * —), (— + —) numerical limit cycle for the conservative force sinh x, sin x, respectively; (— A —), (— B —) analytical limit cycle for the conservative force sinh x, sin x, respectively; (— L —) analytical limit cycle of $\ddot{x} + x = 0.1\,(1 - x^2)\dot{x}$.

For oscillator (38) we find [15]

$$c_1^*(A) = 2[J_1(A) + 3J_3(A)]/A \tag{39}$$

$$c_3^*(A) = -8J_3(A)/A^3 \tag{40}$$

where J_1 and J_3 are Bessel functions. As before $\lambda^*(A)$ is given by equation (34). The C-curve is also plotted in Fig. 1. The intersection point $A_s^2 = 4.236$, $\nu_s = -0.5605$ defines the limit cycle given by equations (27) and (28), with $\omega^2 = 0.422$ and $\sigma^2 = 0.389$ since $c_1^*(A_s) = 0.959$.

In Fig. 4, we compare the above two limit cycles with the limit cycles evaluated numerically for $\varepsilon = 0.1$. As a reference, the limit cycle of $\ddot{x} + x = 0.1(1 - x^2)\dot{x}$ is also plotted.

Finally, for oscillator (38), it is easy to give an estimate of the value of α/β where the limit cycle reaches the saddle points and becomes a separatrix (homoclinic cycle). This happens when $\nu = -1$, i.e. when $c_3^*(A)A^2/c_1^*(A) = -1$, i.e. from equations (39) and (40) when $J_1(A) = J_3(A)$, i.e. for $A = 3.054$. But for $\alpha/\beta = 1$ and $\nu = -1$, one has $Q/T = A_s^2 = 5$. Therefore, from equation (35a) one concludes that there is no oscillatory motion when $\alpha/\beta \geq (3.054)^2/5 \simeq 1.87$. Numerically one obtains $\alpha/\beta > \approx 1.89$ for $\varepsilon = 0.1$.

6. CONCLUSIONS

A new method for the approximate study of a wide class of NLO given by equation (5) has been presented. A cubic oscillator associated with the NLO problem is constructed in such a way that both oscillators are equal in at least their largest harmonics, assuming equation (8) as the solution in a first approximation. An approximate knowledge of the NLO problem was reached studying the associated cubic oscillator. As was shown with some examples, the method is suitable for studying the positions and characteristics of limit cycles and bifurcations, in other words, for studying the essential part of the NLO problem—the topological configuration of its solutions—while sacrificing some other secondary facts (the presence of higher harmonics in the stationary solution, for example). These properties are not at all unexpected: the situation is very similar to that in the linearization methods for NLOs of type (2), i.e. for quasilinear oscillators, see p. 74 of ref. [2].

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