

On a new criterion for evaluating the stability of the limit cycles of perturbed Duffing oscillators

S. Bravo Yuste

Departamento de Física, Facultad de Ciencias, Universidad de Extremadura, 06071 Badajoz, Spain

Abstract

A criterion for finding the orbital asymptotic stability of the limit cycles of the perturbed Duffing oscillators $\ddot{x} + c_1\dot{x} + c_3x^3 + \varepsilon f(x, \dot{x}) = 0$, based on a method of Krylov–Bogoliubov type that uses Jacobi elliptic functions in the approximate solution, is shown to be equivalent to a current well-known criterion, sometimes called the Poincaré criterion. As an example, both criteria are applied to the Duffing oscillator perturbed with the van der Pol term $f(x, \dot{x}) = (\alpha - \beta x^2)\dot{x}$.

1. Introduction

The well-known Poincaré criterion (Minorsky, 1962; Struble, 1962; Mickens, 1981) to determine the stability of the limit cycles of the oscillators

$$\ddot{x} + F(x, \dot{x}) = 0 \tag{1.1}$$

states that a limit cycle (x_s, \dot{x}_s) is stable if

$$\int_0^T F_1(x_s, \dot{x}_s) dt > 0, \tag{1.2}$$

where $F_1(x, \dot{x}) = \partial F(x, \dot{x})/\partial \dot{x}$, and T is the period of $x_s(t)$ and $\dot{x}_s(t)$. (We shall here present the conditions as inequalities that, when satisfied, mean that the limit cycles are stable. The limit cycles are unstable if the contrary inequalities are satisfied. Throughout the paper by stability we mean orbital asymptotic stability.) Unfortunately there are very few nonlinear oscillators in which the limit cycle $x_s(t)$ is known in an exact form. More frequently the limit cycle is given by an approximate expression used in the Poincaré condition (1.2) for evaluating its stability. This approximate expression is calculated by means of some method designed for the task: perturbation methods, the multiple-scale method, the method of harmonic balance, the Krylov–Bogoliubov method, amongst others. However, most methods are strongly limited by only being applicable to weakly nonlinear oscillators:

$$\ddot{x} + c_1\dot{x} + \varepsilon f(x, \dot{x}) = 0, \tag{1.3}$$

where $c_1 > 0$ and ε is a small parameter. The Krylov–Bogoliubov (KB) method has the advantage that it can give not only the stationary solution of the oscillator $x_s(t)$,

but also the transitory oscillations, that is, it can give the complete solution $x(t)$. The KB approximate solution is

$$x(t) = A(t) \cos(\psi(t)), \quad (1.4)$$

where amplitude $A(t)$ and phase $\psi(t)$ are obtained by solving the equations

$$\dot{A} = \frac{\varepsilon}{\omega} \frac{1}{2\pi} \int_0^{2\pi} f(A \cos \psi, -A\omega \sin \psi) \sin \psi d\psi, \quad (1.5)$$

$$\dot{\psi} = \omega + \frac{\varepsilon}{A\omega} \frac{1}{2\pi} \int_0^{2\pi} f(A \cos \psi, -A\omega \sin \psi) \cos \psi d\psi \quad (1.6)$$

with $\omega^2 = c_1$. It is well known (Minorsky, 1962), and indeed easily understood, that the limit cycle is stable if $[d\dot{A}/dA]_s < 0$. (An expression with the subscript s represents the value of this expression at the limit cycle.) It is also well known (Minorsky, 1962; Mickens, 1981) that this KB-based criterion and the Poincaré criterion, though very different in form, are equivalent.

As noted above, most of the methods used to find approximate solutions for nonlinear oscillators are only applicable to weakly nonlinear oscillators. To overcome this limitation several methods have been put forward for a wider class of oscillator. One interesting class is that of the perturbed Duffing oscillators (strongly nonlinear oscillators)

$$\ddot{x} + c_1 x + c_3 x^3 + \varepsilon f(x, \dot{x}) = 0, \quad (1.7)$$

where c_1 and c_3 are arbitrary and ε is a small parameter. The author and colleagues have constructed methods of finding approximate solutions in terms of Jacobi elliptic functions for this oscillator class: a method of harmonic balance (Bravo Yuste and Diaz Bejarano, 1986; Garcia-Margallo, Diaz Bejarano and Bravo Yuste, 1988), a Galerkin method (Bravo Yuste, 1989) and a Krylov–Bogoliubov method (Bravo Yuste and Diaz Bejarano, 1986, 1987, 1989, 1990). Recently, Coppola and Rand (1990) have implemented another method of KB type. This method, which we will call the elliptic KB (EKB) method, is completely equivalent to that given by Bravo Yuste and Diaz Bejarano (1990) when $c_1 = 0$ or $c_3 = 0$. When $c_1 \neq 0$ and $c_3 \neq 0$ the two methods lead to the same equations for the oscillation amplitude. However, the method of Coppola and Rand (1990) is preferable because the phase equation is suitable for the averaging procedure even for $c_1 \neq 0$ and $c_3 \neq 0$. The use of elliptic functions to solve nonlinear oscillators is not new: see the books of Davis (1962), Nayfeh (1973) and Nayfeh and Mook (1979).

An advantage of the EKB method over the generalized method of harmonic balance and Galerkin method is that it gives not only the steady solution (limit cycle) but also the transitory motion. The study of the stability of the limit cycles is then as straightforward as in the usual KB method. The new stability criterion is a simple logical extension to the strongly nonlinear perturbed Duffing oscillators of the criterion deduced from the usual KB method for weakly nonlinear oscillators (Minorsky, 1962; Mickens, 1981).

2. The new stability criterion deduced from the EKB method

The solution of equation (1.7) in the EKB method is given by (Coppola and Rand, 1990)

$$x(t) = A(t) \operatorname{cn}[4K\varphi(t), m(t)] = A(t) \operatorname{cn}[\psi(t), m(t)] \quad (2.1)$$

with

$$\dot{x}(t) = -\omega(t)A(t) \operatorname{sn}[\psi(t), m(t)] \operatorname{dn}[\psi(t), m(t)], \quad \Psi(t) = \int_0^t \omega(s) ds - \phi(t),$$

and where the frequency ω and modulus m are given by

$$\omega^2 = c_1 + c_3 A^2 = c_1(1 + \nu), \quad (2.2)$$

$$m = c_3 A^2 / [2(c_1 + c_3 A^2)] = \nu / [2(1 + \nu)], \quad (2.3)$$

with

$$\nu = c_3 A^2 / c_1 \quad (2.4)$$

the nonlinearity factor. We have denoted by $4K$ the period of the solution: $\operatorname{cn}(\psi + 4K, m) = \operatorname{cn}(\psi, m)$, where

$$K = \begin{cases} K(m) & \text{if } 0 \leq m \leq 1, \\ (1-m)^{-\frac{1}{2}} K(-m/(1-m)) & \text{if } m < 0, \\ \frac{1}{2} m^{-\frac{1}{2}} K(1/m) & \text{if } m > 1, \end{cases} \quad (2.5)$$

and $K(z)$ is the complete elliptic integral of the first kind of modulus z . The functions $\operatorname{cn}(\psi, m)$, $\operatorname{sn}(\psi, m)$ and $\operatorname{dn}(\psi, m)$ are the three basic Jacobi elliptic functions (for more details see (Davis, 1962) or (Abramowitz and Stegun, 1972)). In the following we shall sometimes write the Jacobi elliptic functions without their arguments.

The functions $A(t)$ and $\varphi(t)$ are obtained by solving

$$\dot{A} = (\varepsilon/\omega) \langle f \operatorname{sn} \operatorname{dn} \rangle \equiv g(A), \quad (2.6)$$

$$\dot{\phi} = \frac{\omega}{4K} + \frac{\varepsilon}{4K\omega A} \left\langle f \left[\operatorname{cn} + \frac{1-2m}{1-m} (Z \operatorname{sn} \operatorname{dn} - m \operatorname{cn} \operatorname{sn}^2) \right] \right\rangle, \quad (2.7)$$

where $f \equiv f(A \operatorname{cn}, -A\omega \operatorname{sn} \operatorname{dn})$, $Z = Z(\psi, m) = E(\psi, m) - \psi E/K$ is the zeta elliptic function of Jacobi, $E(\psi, m)$ is the incomplete elliptic integral of the second kind (Abramowitz and Stegun, 1972) and

$$\langle \dots \rangle \equiv \frac{1}{4K} \int_0^{4K} \dots d\psi$$

is the operation of averaging over the period $4K$ of the Jacobi elliptic functions of (2.6) and (2.7). In these expressions K is given by (2.5) and E by

$$E = \begin{cases} E(m) & \text{if } 0 \leq m \leq 1, \\ (1-m)^{\frac{1}{2}} E(-m/(1-m)) & \text{if } m < 0, \\ \frac{1}{2} m^{\frac{1}{2}} [E(1/m) - ((m-1)/m)K(1/m)] & \text{if } m > 1, \end{cases} \quad (2.8)$$

where $E(z)$ is the complete elliptic integral of the second kind of modulus z .

Notice that $c_3 = 0$ for weakly nonlinear oscillators and then, from equation (2.3), one has $m = 0$. But $\text{cn}(\psi, 0) = \cos \psi$, $\text{sn}(\psi, 0) = \sin \psi$, $\text{dn}(\psi, 0) = 1$, $4K(m = 0) = 2\pi$, and then equations (2.6) and (2.7) are the well-known expressions—equations (1.5) and (1.6)—of the usual KB method (Minorsky, 1962; Mickens, 1981). In other words, the KB method is just a special case (weakly nonlinear oscillators) of the EKB method.

From the amplitude equation (2.6), the limit cycle

$$\left. \begin{aligned} x_s(t) &= A_s \text{cn}(\psi_s, m_s), \\ \dot{x}_s(t) &= -\omega_s A_s \text{sn}(\psi_s, m_s) \text{dn}(\psi_s, m_s) \end{aligned} \right\} \quad (2.9)$$

is stable if, as in the KB method,

$$[d\dot{A}/dA]_s \equiv [dg/dA]_s < 0. \quad (2.10)$$

The Poincaré criterion in our case is

$$\int_0^{4K} F_1[A_s \text{cn}(\psi_s, m_s), -\omega_s A_s \text{sn}(\psi_s, m_s) \text{dn}(\psi_s, m_s)] dt > 0, \quad (2.11)$$

where $F_1 = \varepsilon \partial f / \partial \dot{x}$. At first sight the new criterion based on the EKB method, equation (2.10), and the Poincaré criterion, equation (2.11), seem very different. However, we shall show in the next section that the two criteria applied to the van der Pol–Duffing oscillator

$$\ddot{x} + c_1 \dot{x} + c_3 x^3 = \varepsilon(\alpha - \beta x^2)\dot{x}, \quad (2.12)$$

with $\alpha/\beta > 0$, lead to identical results. This very interesting oscillator appears in the analysis of flow-induced oscillators (Holmes, 1977) and lasers with a saturable absorber (Antoranz and Rubio, 1988). A study of this equation using differentiable dynamics can be found in (Holmes and Rand, 1980) or in (Guckenheimer and Holmes, 1983).

That the two criteria lead to the same results for this oscillator is not accidental: in section 4 we shall show that the two criteria are equivalent.

3. An example: the van der Pol–Duffing oscillator

The equation of the van der Pol–Duffing oscillator is given by (2.12). By using equation (2.6) we obtain

$$\dot{A} \equiv g(A) = \varepsilon[\alpha \langle \text{sn}^2 \text{dn}^2 \rangle - \beta \langle \text{sn}^2 \text{cn}^2 \text{dn}^2 \rangle A^2] = \varepsilon(\tilde{\alpha} - \tilde{\beta} A^2) = \varepsilon \tilde{\alpha}(1 - A^2/G) \quad (3.1)$$

with (Byrd and Friedman, 1971)

$$\langle \text{sn}^2 \text{dn}^2 \rangle = \frac{1}{3m} \frac{(2m-1)E + (1-m)K}{K}, \quad (3.2)$$

$$\langle \text{sn}^2 \text{cn}^2 \text{dn}^2 \rangle = (1/15m^2)[m_1(m-2)K + 2(m^2 + m_1)E]/K, \quad (3.3)$$

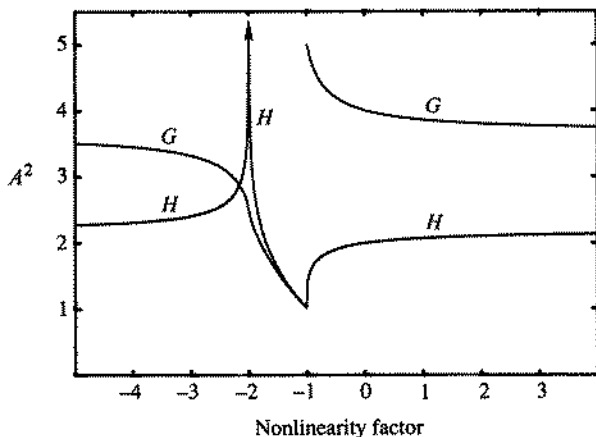


Fig. 1. The functions G and H versus the nonlinearity factor v . With the Poincaré criterion, when $\varepsilon > 0$ the limit cycle with nonlinearity factor v_s will be stable (unstable) if $G(v_s)$ is greater (smaller) than $H(v_s)$. With the new criterion, when $\varepsilon > 0$ the limit cycle with nonlinearity factor v_s will be stable (unstable) if the line tangential to the curve G at v_s passes above (below) the origin. The limit cycle with $v_s = -2.164$ and $A_s^2 = 2.877$ (for which G is equal to H and the line tangent to G at v_s passes through the origin) is semistable. The two criteria thus lead to identical results

$m_1 = 1 - m$, $\tilde{\alpha} = \alpha \langle \text{sn}^2 \text{dn}^2 \rangle$, $\tilde{\beta} = \beta \langle \text{sn}^2 \text{cn}^2 \text{dn}^2 \rangle$ and

$$G(m) = \tilde{\alpha}/\tilde{\beta}. \quad (3.4)$$

In these relations K is given by (2.5) and E by (2.8).

From equation (3.1) $\dot{A} = 0$ when $A^2 = G(A)$. Thus, for amplitudes that satisfy this equation, the van der Pol–Duffing oscillator has a limit cycle. Instead of solving this equation it is preferable to solve the system

$$A^2 = G(v), \quad A^2 = c_1 v / c_3. \quad (3.5)$$

The last equation (3.5)₂ comes from the definition of the nonlinearity factor v given by relation (2.4). Notice that the value v_s for which the system (3.6) is satisfied gives, by means of equations (3.5)₂, (2.2), (2.3) and (2.9), the limit cycle of the oscillator. The function $G(v)$ for $\alpha/\beta = 1$ is shown in Fig. 1. To obtain the amplitude A_s and the value v_s of the limit cycle for a given oscillator with some determined coefficients c_1 and c_3 , we simply plot the straight line $A^2 = c_1 v / c_3$ —equation (3.5)₂—and the curve $G(v)$ —equation (3.5)₁—so that their intersection gives v_s and A_s .

We shall now study the stability of these limit cycles by using the criterion of Poincaré and the new criterion based on the EKB method. For the van der Pol–Duffing oscillator $F_1(x, \dot{x}) = -\varepsilon(\alpha - \beta x^2)$, evaluating the integral (Byrd and Friedman, 1971) of equation (2.11) of the Poincaré criterion, one finds, for $\varepsilon > 0$ ($\varepsilon < 0$) that the limit cycles are stable (unstable) if $G > H$, where

$$H = \frac{\alpha}{\beta} \frac{Km}{E - m_1 K} \quad \text{for } 0 \leq m \leq 1 \quad (3.6a)$$

with $K = K(m)$ and $E = E(m)$;

$$H = \frac{\alpha}{\beta} \frac{K\sigma^2}{K - E} \quad \text{for } m \leq 0 \quad (3.6b)$$

with $K = K(\sigma^2)$, $E = E(\sigma^2)$, $\sigma^2 = m/(1 - m)$; and

$$H = \frac{\alpha}{\beta} \frac{K}{E} \quad \text{for } m \geq 1 \quad (3.6c)$$

with $K = K(\eta^2)$, $E = E(\eta^2)$ and $\eta^2 = 1/m$. A plot of the function $H(v)$ for $\alpha/\beta = 1$ is shown in Fig. 1. The functions G and H are equal only for $v_s = v_c \equiv -2.164$. In the following we assume that $\varepsilon > 0$. From the Poincaré criterion and from Fig. 1 one sees that limit cycles are stable if $v_s < v_c$ or $v_s > -1$, and unstable if $v_c < v_s < -1$. The limit cycle with $v_s = v_c$ is semistable.

Using the new criterion based on the EKB method we find from (3.1) and (2.10) that the limit cycles are stable if

$$\left(\frac{d}{dA} g(A) \right)_s = \left(\varepsilon \tilde{\alpha} 2A \frac{d}{dA^2} (1 - A^2/G) \right)_s < 0. \quad (3.7)$$

In the following we assume that $\alpha > 0$. As $\langle (\text{sn dn})^2 \rangle > 0$ for all v (except $v = -2$ and $v = -1$), after some simple operations we find that the limit cycle is stable if

$$\frac{d}{dA^2} G < 1 \quad (3.8)$$

or, in terms of the nonlinearity factor ν , if

$$\frac{c_3}{c_1} \frac{d}{d\nu} G < 1. \quad (3.9)$$

Therefore, when $c_3/c_1 > 0$, the limit cycle of amplitude A_s is stable if $dG/d\nu < c_1/c_3$, and, when $c_3/c_1 < 0$, the limit cycle of amplitude A_s is stable if $dG/d\nu > c_1/c_3$. Referring to Fig. 1 these results can be reformulated as follows. If the tangent to the curve G at the point (v_s, A_s^2) crosses the ordinate axis above the origin $(0, 0)$, then the limit cycle of amplitude A_s is stable, that is, the limit cycles with $v_s < v_c$ or $v_s > -1$ are stable; if the tangent crosses below the origin the limit cycle is unstable, that is, the limit cycles with $v_c < v_s < -1$ are unstable. For $v_s = v_c$ the tangent passes through the origin and the limit cycle is semistable. Therefore, one finds that the Poincaré criterion and the new EKB-based criterion lead to identical results and conclusions for the van der Pol–Duffing oscillator. In the next section we shall show that this is valid for any oscillator.

4. Proof of the equivalence of the two criteria

We shall show the equivalence of the present EKB based criterion and the Poincaré criterion by demonstrating that

$$\left(\frac{\partial}{\partial A} g(A) \right)_s = -\varepsilon \left(\frac{1}{4K} \int_0^{4K} \frac{\partial}{\partial \tilde{x}} f(A \text{ cn}, -\omega A \text{ sn dn}) d\psi \right)_s. \quad (4.1)$$

From the expression for $g(A)$ of equation (2.6), one has

$$\frac{\partial}{\partial A} g(A) = \varepsilon \left[\frac{d}{dA} \left(\frac{1}{\omega 4K} \right) \right] \int_0^{4K} f \operatorname{sn} \operatorname{dn} d\psi + \varepsilon \frac{1}{\omega 4K} \frac{d}{dA} \int_0^{4K} f \operatorname{sn} \operatorname{dn} d\psi, \quad (4.2)$$

where we simply write f for $f(A \operatorname{cn}, -\omega A \operatorname{sn} \operatorname{dn})$. The first integral, evaluated for the limit cycle, is zero because it is proportional to $g(A)$ and $g(A)$ is zero for $A = A_s$. The second integral is

$$\frac{d}{dA} \int_0^{4K} f \operatorname{sn} \operatorname{dn} d\psi = \int_0^{4K} \frac{d}{dA} [f \operatorname{sn} \operatorname{dn}] d\psi + 4 \frac{dK}{dA} f \operatorname{sn}(4K, m) \operatorname{dn}(4K, m). \quad (4.3)$$

But $\operatorname{sn}(4K, m) = 0$ and $\partial \operatorname{cn} / \partial \psi \equiv \operatorname{cn}_\psi = -\operatorname{sn} \operatorname{dn}$. Then (4.2) becomes

$$\left(\frac{\partial}{\partial A} g(A) \right)_s = -\varepsilon \left(\frac{1}{\omega} \frac{1}{4K} \int_0^{4K} \operatorname{cn}_\psi \frac{df}{dA} d\psi + \frac{1}{\omega} \frac{1}{4K} \int_0^{4K} f \operatorname{cn}_{\psi A} d\psi \right)_s. \quad (4.4)$$

Changing the derivative order of $\operatorname{cn}_{\psi A}$ in the last integral and integrating by parts one finds that

$$\int_0^{4K} f \frac{d}{dA} \operatorname{cn}_\psi d\psi = \int_0^{4K} f \frac{d}{d\psi} \operatorname{cn}_A d\psi = [f \operatorname{cn}_A]_0^{4K} - \int_0^{4K} \operatorname{cn}_A \frac{df}{d\psi} d\psi. \quad (4.5)$$

The bracket is zero since f and cn_A are periodic functions of period $4K$. Therefore (4.4) becomes

$$\left(\frac{\partial}{\partial A} g(A) \right)_s = -\varepsilon \left(\frac{1}{\omega} \frac{1}{4K} \int_0^{4K} \operatorname{cn}_\psi \frac{df}{dA} d\psi - \frac{1}{\omega} \frac{1}{4K} \int_0^{4K} \operatorname{cn}_A \frac{df}{d\psi} d\psi \right)_s. \quad (4.6)$$

But

$$\frac{df}{d\psi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \psi} + \frac{\partial f}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \psi} \quad (4.7)$$

and as $x = A \operatorname{cn} \psi$ and $\dot{x} = A\omega \operatorname{cn}_\psi = -A\omega \operatorname{sn} \operatorname{dn}$ at the limit cycle, one has

$$\frac{df}{d\psi} = \frac{\partial f}{\partial x} A \operatorname{cn}_\psi + \frac{\partial f}{\partial \dot{x}} A\omega \operatorname{cn}_{\psi\psi}. \quad (4.8)$$

Also

$$\frac{df}{dA} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial A} + \frac{\partial f}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial A} \quad (4.9)$$

and then

$$\frac{df}{dA} = \frac{\partial f}{\partial x} [\operatorname{cn} + A \operatorname{cn}_A] + \frac{\partial f}{\partial \dot{x}} \left[\frac{\partial}{\partial A} (A\omega) \operatorname{cn}_\psi + A\omega \operatorname{cn}_{\psi A} \right]. \quad (4.10)$$

Putting (4.8) and (4.10) into (4.6) one obtains, after some algebra,

$$\begin{aligned} \left(\frac{\partial}{\partial A} g(A) \right)_s = & -\varepsilon \left(\frac{1}{\omega} \frac{1}{4K} \right)_s \left\{ \int_0^{4K} \frac{df}{dx} \operatorname{cn} \operatorname{cn}_\psi d\psi \right. \\ & \left. + \int_0^{4K} \frac{\partial f}{\partial \dot{x}} \left(\frac{\partial(A\omega)}{\partial A} (\operatorname{cn}_\psi)^2 + A\omega [\operatorname{cn}_{\psi A} \operatorname{cn}_\psi - \operatorname{cn}_{\psi\psi} \operatorname{cn}_A] \right) d\psi \right\}_s. \end{aligned} \quad (4.11)$$

But

$$\operatorname{cn}_{\psi A} \operatorname{cn}_{\psi} - \operatorname{cn}_{\psi\psi} \operatorname{cn}_A = m_A [\operatorname{cn}_{\psi m} \operatorname{cn}_{\psi} - \operatorname{cn}_{\psi\psi} \operatorname{cn}_m] = -m_A \operatorname{sn}^4/2 \quad (4.12)$$

and

$$\left. \begin{aligned} \operatorname{sn}^4 &= 1 - 2\operatorname{cn}^2 + \operatorname{cn}, \\ (\operatorname{cn}_{\psi})^2 &= 1 - m + (2m - 1)\operatorname{cn}^2 - m \operatorname{cn}^4, \\ \frac{\partial(\omega A)}{\partial A} &= (2m + 1)\omega, \\ m_A &= 2m(1 - 2m)/A. \end{aligned} \right\} \quad (4.13)$$

Then, using (4.12) and (4.13), it is not too difficult to deduce that

$$\frac{\partial(A\omega)}{\partial A} (\operatorname{cn}_{\psi})^2 + A\omega [\operatorname{cn}_{\psi A} \operatorname{cn}_{\psi} - \operatorname{cn}_{\psi\psi} \operatorname{cn}_A] = \omega + \omega \operatorname{cn}_{\psi\psi} \operatorname{cn}. \quad (4.14)$$

Therefore equation (4.11) becomes

$$\left(\frac{\partial}{\partial A} g(A) \right)_s = -\varepsilon \left\{ \frac{1}{4K} \int_0^{4K} \frac{df}{d\dot{x}} d\psi - \frac{1}{\omega} \frac{1}{4K} \int_0^{4K} \left[\frac{\partial f}{\partial x} \operatorname{cn}_{\psi} + \frac{\partial f}{\partial \dot{x}} \omega \operatorname{cn}_{\psi\psi} \right] \operatorname{cn} d\psi \right\}_s. \quad (4.15)$$

But the term in the brackets is equal to $[\partial f/\partial \psi]/A$ —see equation (4.8)—and then

$$\left(\frac{\partial}{\partial A} g(A) \right)_s = -\varepsilon \left(\frac{1}{4K} \int_0^{4K} \frac{df}{d\dot{x}} d\psi + \frac{1}{A\omega} \frac{1}{4K} \int_0^{4K} \frac{\partial f}{\partial \psi} \operatorname{cn} d\psi \right)_s. \quad (4.16)$$

Integrating the last integral by parts, one finds that

$$\left(\int_0^{4K} \frac{\partial f}{\partial \psi} \operatorname{cn} d\psi \right)_s = \left([f \operatorname{cn}]_0^{4K} - \int_0^{4K} f \operatorname{cn}_{\psi} d\psi \right)_s = 0. \quad (4.17)$$

This is because (i) the bracketed term is zero since, in the limit cycle, f and cn are periodic functions of period $4K$, and (ii) the second integral evaluated for the limit cycle is zero because this integral is proportional to $g(A)$. Therefore, since the second integral of (4.16) is zero for the limit cycle, the relation (4.1) is verified.

5. Conclusions

In this paper we have described a new criterion for evaluating the stability of limit cycles for a class of strong nonlinear oscillators: the perturbed Duffing oscillators $\ddot{x} + c_1 x + c_3 x^3 + \varepsilon f(x, \dot{x}) = 0$. This new criterion is based on a method of Krylov–Bogoliubov type that uses Jacobi elliptic functions in the approximate solution. We have applied this new criterion and the well-known criterion of Poincaré to the van der Pol–Duffing oscillator obtaining identical results. We saw that this is not accidental: in section 4 we proved in a general form that the two criteria are equivalent for any oscillator.

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