Generalized Bohr-Sommerfeld rule for quartic oscillators

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The generalized Bohr-Sommerfeld quantization condition up to the ninth-order approximation is given explicitly, in terms of complete elliptic integrals, for two kinds of quartic oscillators $V(z)=v_2z^2+v_4z^4$: (i) for the quartic double barrier, where $v_2>0$ and $v_4<0$, and the energy is below the potential maximum; and (ii) for the quartic double well, where $v_2<0$ and $v_4>0$. Resonance energies for some quartic double barriers and energy levels for some quartic double wells were evaluated by solving the quantization condition numerically. In some cases the results are very good.

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I. INTRODUCTION

The one-dimensional Schrödinger equation for a quartic oscillator potential

$$\frac{d^2\psi}{dz^2}+R(z)\psi=0 \quad (1.1)$$

with

$$R(z)=\frac{2\mu}{\hbar^2}[\mathcal{E}-V(z0] , \quad (1.2)$$

$$V(z)=v_2z^2+v_4z^4 \quad (1.3)$$

(v_2 and v_4 constants) has a considerable interest in many branches of physics [1] and a great variety of methods have been used to investigate the eigenvalues and other properties. The quantum quartic oscillator has in fact been the major proving ground for tests of approximation methods in quantum mechanics. Quantum oscillators with $v_2\geq 0$ and $v_4\geq 0$ (quartic wells) and with $v_2<0$ and $v_4>0$ (quartic double wells) have been the most investigated [2]. By far the least studied quartic oscillators are the double barriers: $v_2>0$ and $v_4<0$ with energy below the maximum potential, $\mathcal{E}<V_{\max}=-v_2^2/(4v_4)$ [3–5].

An old and well-known procedure for finding energy levels of quartic wells, energy resonances of quartic double barriers, and unsplit energy levels of quartic double wells, is the Bohr-Sommerfeld (BS) rule [4,6]. It has been known, since the pioneer work of Dunham [7], that this quantization condition can be interpreted as the particular case of a more general quantization condition in which only the first-order term is kept. This more general condition can be obtained from the JWKB method of higher order, but here I will use the quantization condition obtained from the closely related phase-integral method of Fröman and Fröman [8], that, for several reasons [9,10], is preferable to the JWKB method. In the terminology of Lakshmanan, Karlsson, and Fröman [11], this quantization condition is called the generalized Bohr-Sommerfeld (GBS) rule. An advantage of the GBS rule with respect to other methods (e.g., perturbation or variational methods) is that its application is computationally straightforward and not time consuming if higher-order terms are known. However, it is also true that the GBS rule cannot, in general, give the energy values with arbitrary accuracy because energies obtained by means of successive higher-order approximations only constitute an asymptotic expansion in the expansion parameter h. Perhaps the major disadvantage of the GBS procedure is that higher-order terms involve the evaluation of very difficult integrals. In 1981 Lakshmanan, Karlsson, and Fröman [11] and, independently, Kesarwani and Varshni [12] were able to evaluate the integrals of the first few terms of the GBS rule for the single quartic well and so obtain very accurate eigenenergies (generally far better than those found by the BS rule). However, to the best of my knowledge, similar work has never been reported for the important cases of the quartic double barrier and quartic double well. The principal aim of this paper is to remedy this situation. I will give explicit analytical expressions of the first five GBS terms for the double barrier and double-well quartic potentials. These expressions are obtained by applying the transformation properties of the elliptic functions with respect to the elliptic modulus to the relationship found by Lakshmanan, Karlsson, and Fröman [11] in the quartic well.

The structure of the paper is as follows. Section II gives a short presentation of the phase-integral method and a summary of expressions and results given in Refs. [11] and [12] for the quartic well. In Sec. III it is shown how to exploit these expressions to evaluate the first five terms of the GBS quantization condition for quartic double barriers. Results obtained using this rule are then compared with those reported by Drummond [3]. In Sec. IV, the first five terms of the GBS rule for the quartic double well are given, and the general expressions obtained are applied to two particular cases and the results compared with those reported by Hodgson and Varshni [13]. Finally, some concluding remarks are offered in Sec. V.

II. GBS RULE FOR THE QUARTIC WELL ($v_2\geq 0$, $v_4\geq 0$)

This section is mainly a summary of expressions and results given in Ref. [11]. Interested readers will find more details there. I will follow the notation of this reference as closely as possible. In the phase-integral method of Fröman and Fröman [8] the exact solution of Eq. (1.1) is written as

$$\psi(z)=a_1(z)f_1(z)+a_2(z)f_2(z)\ , \tag{2.1}$$

where the phase-integral functions $f_1(z)$ and $f_2(z)$ are

$$f_1(z)=q^{-1/2}\exp\left[+i\int^z q(z)dz\right]\ , \tag{2.2a}$$

$$f_2(z)=q^{-1/2}\exp\left[-i\int^z q(z)dz\right]\ . \tag{2.2b}$$

The function $q(z)$ is determined by imposing that

$$\psi(z)=q^{-1/2}\exp\left[\pm i\int^z q(z)dz\right] \tag{2.3}$$

satisfies Eq. (1.1), i.e., putting (2.3) into (1.1), $q(z)$ must satisfy

$$q^{-3/2}\frac{d^2}{dz^2}q^{-1/2}+R(z)/q^2-1=0\ . \tag{2.4}$$

This equation is strictly equivalent to the original Schrödinger equation, Eq. (1.1). Let $Q(z)$ be an unspecified approximate solution of (2.4). Then the quantity

$$\epsilon_0=Q^{-3/2}(z)\frac{d^2}{dz^2}Q^{-1/2}(z)+\frac{R(z)-Q^2(z)}{Q^2} \tag{2.5}$$

will be small compared with unity. With the exact solution $q(z)$ written as

$$q(z)=Q(z)g(z) \tag{2.6}$$

Fröman and Fröman proved (see, for example, Ref. [14]) that $g(z)$ can be expressed as an asymptotic series:

$$g(z)=\sum_{i=0}^{N}Y_{2i}\ , \tag{2.7}$$

where

$$Y_0=Z_0\ , \tag{2.8a}$$

$$Y_2=Z_2\ , \tag{2.8b}$$

$$Y_4=Z_4-\frac{1}{8}\frac{d^2\epsilon_0}{d\zeta^2}\ , \tag{2.8c}$$

$$Y_6=Z_6+\frac{1}{32}\frac{d}{d\zeta}\left[6\epsilon_0\frac{d\epsilon_0}{d\zeta}+\frac{d^3\epsilon_0}{d\zeta^3}\right]\ , \tag{2.8d}$$

etc., with

$$Z_0=1\ , \tag{2.9a}$$

$$Z_2=\tfrac{1}{2}\epsilon_0\ , \tag{2.9b}$$

$$Z_4=-\tfrac{1}{8}\epsilon_0^2\ , \tag{2.9c}$$

$$Z_6=\frac{1}{32}\left[2\epsilon_0^3-\left[\frac{d\epsilon_0}{d\zeta}\right]^2\right]\ , \tag{2.9d}$$

etc., and

$$\zeta=\int^z Q(z)dz\ . \tag{2.10}$$

There exists a recursion relation which allows one to find any term Y_{2n} [14]. (Expressions up to Y_8 are given in [15] and up to Y_{20} in [16].) The $N+1$-term (or $2N+1$-order) approximate solution of (2.4) is

$$q(z)=Q(z)\sum_{i=0}^{N}Y_{2i}\ . \tag{2.11}$$

Hereafter, a particular choice (in fact, the simplest and most usual choice) of the as yet unspecified function $Q(z)$ that generates the phase-integral approximation is used, namely,

$$Q^2(z)=R(z)\ . \tag{2.12}$$

At this point, some comments about the notation may be appropriate. In this paper the most recent notation of Fröman and Fröman [10] is used, $Q(z)$ and $R(z)$ denoting the functions represented by $Q_{\mathrm{mod}}(z)$ and $Q(z)$, respectively, in their previous papers [9,11,14,15,17].

In Fröman and Fröman's phase-integral method the quantization condition of order $2N+1$ for a single well is given by [17]

$$\tfrac{1}{2}\int_\Gamma Q(z)\sum_{i=0}^{N}Y_{2i}dz+\Delta=(n+\tfrac{1}{2})\pi\ ,\quad n=0,1,2,\ldots\ , \tag{2.13}$$

where

$$\Delta=\arg\left[\frac{F_{22}(z,+\infty)}{F_{22}(z,-\infty}\right]\ , \tag{2.14}$$

F_{22} being the fourth element of the F matrix defined in [18], and Γ a closed loop in the complex z plane enclosing both calssical turning points but no other zeros or singularities of $Q^2(z)=R(z)$ [see Fig. 1(a)]. Neglecting the term Δ, we get the GBS rule of $N+1$ terms (order $2N+1$):

$$\sum_{i=0}^{N}T_i=\tfrac{1}{2}\int_\Gamma Q(z)\sum_{i=0}^{N}Y_{2i}dz=(n+\tfrac{1}{2})\pi\ ,\quad n=0,1,2,\ldots\ . \tag{2.15}$$

Notice that for $N=0$ we retrieve the usual well-known BS rule.

The mathematical structure of (2.8) has the general form $Y_{2n}=Z_{2n}+dU_{2n}/d\zeta$, and therefore (2.15) is equivalent to

$$\sum_{i=0}^{N}T_i=\tfrac{1}{2}\int_\Gamma\sum_{i=0}^{N}Z_{2i}d\zeta=(n+\tfrac{1}{2})\pi\ ,\quad n=0,1,2,\ldots\ . \tag{2.16}$$

As is shown in detail in [11], defining

$$\tau=\int^z\frac{dz}{Q(z)}\ , \tag{2.17}$$

and after some partial integrations, one can write the terms that appear in (2.16) equivalently as

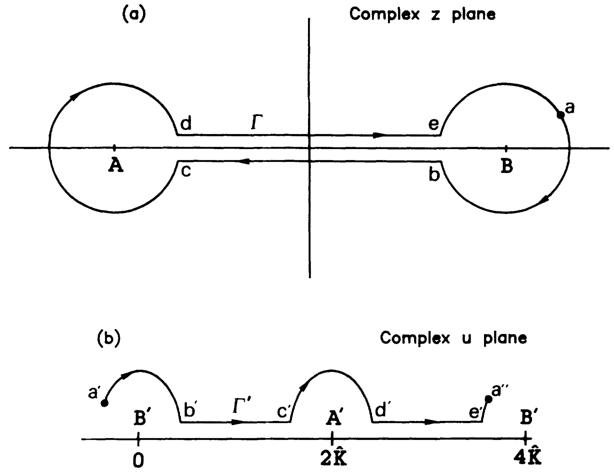

FIG. 1. (a) The integration path Γ in the complex z plane, where A and B are the classical turning points. (b) The integration path Γ' in the complex u plane. The points A' and B' correspond to the classical turning points A and B and the points a', b', c', d', e', and a'' are points of Γ' corresponding to points a, b, c, d, e, and a, respectively, on the path Γ in the complex z plane [z and u are connected by Eq. (2.25) for the quartic well, by Eq. (3.3) for the quartic double barrier, and by Eq. (4.3) for the quartic double well].

$$T_0=\frac{1}{2}\int_\Gamma Q^2 d\tau \ , \tag{2.18a}$$

$$T_1=-\frac{1}{48}\int_\Gamma \frac{1}{Q^3}\frac{d^2Q}{d\tau^2}d\tau \ , \tag{2.18b}$$

$$T_2=\frac{1}{768}\int_\Gamma \left[\frac{35}{Q^9}\left[\frac{dQ}{d\tau}\right]^2\frac{d^2Q}{d\tau^2}-\frac{12}{Q^8}\left[\frac{d^2Q}{d\tau^2}\right]^2\right]d\tau \ , \tag{2.18c}$$

etc.

From Eq. (2.17),

$$\frac{dz}{d\tau}=Q \ , \tag{2.19}$$

and so

$$\frac{d^2z}{d\tau^2}=\frac{1}{2}\frac{d(Q^2)}{dz}=\frac{1}{2}\frac{dR}{dz} \ . \tag{2.20}$$

This expression will be known as "the classical equation of motion," as is readily understood if we define

$$t=\frac{\mu\tau}{\hbar} \ , \tag{2.21}$$

so that Eq. (2.20) adopts the usual form

$$\mu\frac{d^2z}{dt^2}=-\frac{dV(z)}{dz} \ . \tag{2.22}$$

For all three quartic oscillators (single well, double well, and double barrier) one has

$$R(z)=Q^2(z)=a-bz^2-cz^4 \tag{2.23}$$

with

$$a=2\mu\mathcal{E}/\hbar^2 \ , \tag{2.24a}$$

$$b=2\mu v_2/\hbar^2 \ , \tag{2.24b}$$

$$c=2\mu v_4/\hbar^2 \ . \tag{2.24c}$$

As is well known [19], the general solution of the "classical equation of motion" for *all* three quartic oscillators, i.e., the general solution of Eq. (2.20) with $R(z)=Q^2(z)$ given by (2.23) can be expressed in terms of the cn Jacobian elliptic function:

$$z=A\,\mathrm{cn}(u,k^2)\equiv A\,\mathrm{cn}u \tag{2.25}$$

with

$$u=\gamma\tau+\delta \tag{2.26}$$

and

$$\gamma=(b+2cA^2)^{1/2} \ . \tag{2.27}$$

Here, A is the constant amplitude and δ is a constant phase. The elliptic modulus k is given by

$$k^2=\frac{cA^2}{b+2cA^2} \ . \tag{2.28}$$

The complementary elliptic modulus, k', is defined by

$$k'^2=1-k^2 \ . \tag{2.29}$$

The energy $\mathcal{E}$ and amplitude A are related by

$$\mathcal{E}=V(A)=v_2A^2+v_4A^4 \ . \tag{2.30}$$

By using well-known formulas [20,21], it is not difficult to write Q and its derivatives that appear in Eqs. (2.18) in terms of Jacobian elliptic functions

$$Q=\frac{dz}{d\tau}=\frac{d}{d\tau}(A\,\mathrm{cn}u)=-A\gamma\,\mathrm{sn}u\,\mathrm{dn}u \ , \tag{2.31a}$$

$$\frac{dQ}{d\tau}=-A\gamma^2(\mathrm{dn}^2u-k^2\mathrm{sn}^2u)\mathrm{cn}u \ , \tag{2.31b}$$

etc. In this way the terms T_i, Eqs. (2.18), become

$$T_0=\frac{\kappa}{8}\int_{\Gamma'}\mathrm{sn}^2u\,\mathrm{dn}^2u\,du \ , \tag{2.32a}$$

$$T_1=\frac{1}{12\kappa}\int_{\Gamma'}\frac{1+4k^2-6k^2\mathrm{sn}^2u}{\mathrm{sn}^2u\,\mathrm{dn}^2u}du \ , \tag{2.32b}$$

etc. [11], with

$$\kappa=4\gamma A^2 \ . \tag{2.33}$$

Here, Γ' is the integration path in the complex u plane that corresponds, through Eq. (2.25), to the Γ integration path in the complex z plane [see Figs. 1(a) and 1(b)]. The path Γ' runs from u_0 (the image of the arbitrary starting point, a, of the path Γ) to $u_0+4\hat{K}$, where $\hat{K}\equiv K(k^2)$, because $4K(k^2)$ is the period in u of the classical solution (2.25). The function K is the complete elliptic integral of the first kind.

In their article Lakshmanan, Karlsson, and Fröman showed that the integrals T_i can be (not trivially) decomposed in such a way that their values depend linearly on three quantities only:

$$\hat{B}_0=\hat{I}_0=\int_{\Gamma'} du \ , \tag{2.34}$$

$$\hat{B}_2=\int_{\Gamma'} \frac{du}{\mathrm{sn}^2 u} \ , \tag{2.35}$$

$$\hat{I}_2=\int_{\Gamma'} \frac{du}{\mathrm{dn}^2 u} \ . \tag{2.36}$$

For the quartic well ($v_2 \geq 0$, $v_4 \geq 0$), the elliptic modulus of the classical solution satisfies $0 \leq k^2 \leq \frac{1}{2}$, and it is then not difficult [21] to evaluate the integrals (2.34)–(2.36):

$$\hat{B}_0=\hat{I}_0=4\hat{K} \ , \tag{2.37}$$

$$\hat{B}_2=4(\hat{K}-\hat{E}) \ , \tag{2.38}$$

$$\hat{I}_2=\frac{4}{k'^2}\hat{E} \ , \tag{2.39}$$

where

$$\hat{K} \equiv K(k^2) \tag{2.40}$$

and

$$\hat{E} \equiv E(k^2) \ . \tag{2.41}$$

The function E is the complete elliptic integral of the second kind.

After some lengthy algebra, Lakshmanan, Karlsson, and Fröman found explicit expressions for the first four terms of the GBS rule. The fifth term was obtained by Kesarwani and Varshni in Ref. [12]. For the sake of completeness, since these expressions will be used in Secs. III and IV and also because Refs. [11] and [12] use different notations, I will now write out all five terms:

$$T_i=P_i(k^2)\frac{\hat{K}-\hat{E}}{k^2}+(-1)^i P_i(k'^2)\frac{\hat{E}}{k'^2} \ , \tag{2.42}$$

where $\hat{K}$ and $\hat{E}$ are given by Eqs. (2.40) and (2.41) and where

$$P_0(x)=\frac{\kappa k'^2}{6} \ , \tag{2.43a}$$

$$P_1(x)=\frac{k^2}{3\kappa}(1+4x) \ , \tag{2.43b}$$

$$P_2(x)=-\frac{k^2}{45\kappa^3 k'^4}(56-153x+285x^2-9320x^3+32\,400x^4-37\,632x^5+14\,336x^6) \ , \tag{2.43c}$$

$$P_3(x)=\frac{2k^2}{315\kappa^5 k'^8}(3968-12\,952x+19\,393x^2+4342x^3-222\,227x^4+17\,667\,524x^5-141\,913\,296x^6 +459\,879\,744x^7-766\,823\,424x^8+699\,572\,224x^9-333\,185\,024x^{10}+65\,011\,712x^{11}) \ , \tag{2.43d}$$

$$P_4(x)=-\frac{k^2}{315\kappa^7 k'^{12}}(390\,144-1\,652\,352x+2\,933\,792x^2-2\,566\,163x^3-2\,812\,943x^4-3\,313\,449x^5 +315\,349\,451x^6-42\,426\,225\,088x^7+563\,786\,106\,016x^8-3\,172\,896\,651\,264x^9 +9\,945\,571\,750\,656x^{10}-19\,344\,051\,593\,216x^{11}+24\,405\,366\,407\,168x^{12}-20\,085\,735\,424\,000x^{13} +10\,440\,335\,687\,680x^{14}-3\,119\,354\,216\,448x^{15}+409\,095\,634\,944x^{16}) \ . \tag{2.43e}$$

By using Eqs. (2.42) and (2.43), one can write the GBS quantization condition, Eq. (2.15), in an explicit analytical form up to the ninth-order (five-term) approximation. For a given quartic well (v_2 and v_4 fixed), the left-hand side of this equation is only a function of the amplitude A via the elliptic modulus k^2 [see Eq. (2.28)]. I will denote by A_n the amplitude that satisfies this equation when the quantum number is n. The value of A_n is determined by solving the equation numerically. The energy for the state with quantum number n is obtained from Eq. (2.30): $\mathcal{E}_n=V(A_n)$. As we pointed out in Refs. [11] and [12], the evaluation of $\mathcal{E}_n$ by solving the quantization condition Eq. (2.15) numerically is computationally straightforward and not time consuming. The interested reader can find a careful analysis of the results obtained using the GBS rule for the quartic well in these references.

III. GBS RULE FOR THE QUARTIC DOUBLE BARRIER ($v_2>0$, $v_4<0$, $\mathcal{E}<V_{\max}$)

For symmetric double barriers the phase-integral quantization condition of $N+1$ terms (order $2N+1$) is the same [see Eq. (32) of [22]] as the phase-integral quantization condition of $N+1$ terms in a single well, Eq. (2.13), but now with

$$\Delta=\frac{\pi}{2}-\arg[F_{12}(-\infty,0)]-\arg[F_{22}(-\infty,0)] \ , \tag{3.1}$$

F_{12} and F_{22} being elements of the F matrix defined in [18]. The integration contour Γ is now a closed loop in the complex z plane enclosing the two *inner* classical turning points but no other zeros or singularities of $Q^2(z)$

[see Fig. 1(a)]. The quantity Δ is usually negligible for energies that are not close to the potential maximum $V_{\max}$. Neglecting this term, we have the GBS rule for symmetric double barriers

$$\sum_{i=0}^{N} T_i=\frac{1}{2}\int_{\Gamma}\sum_{i=0}^{N} Z_{2i}dz=(n+\tfrac{1}{2})\pi\ , \quad n=0,1,2,\ldots\ , \tag{3.2}$$

where use has been made of the equality between Eqs. (2.15) and (2.16). Notice that the GBS rule is formally the same for the single well and for the double barrier (we will see in the next section that it is also the same for the double well).

Using the same definitions as in Sec. II, it is clear that Eqs. (2.18)–(2.24) are also valid for the quartic double barrier. The general solution for the classical equation of motion, Eq. (2.20), is

$$z=A\ \mathrm{cn}(u,k^2)=A\ \mathrm{cd}(u/\sigma',\sigma^2)\ , \tag{3.3}$$

where u, γ, and k^2 are given by Eqs. (2.26)–(2.28), respectively. All symbols have the same meaning as in Sec. II. Now k^2 is negative and, therefore, using the transformation properties of the Jacobian elliptic functions with respect to the negative elliptic modulus [20], the classical solution can be conveniently expressed in terms of the Jacobian elliptic function cd, whose elliptic modulus is given by

$$\sigma^2=-\frac{k^2}{1-k^2}\ . \tag{3.4}$$

The complementary elliptic modulus, σ', is defined by $\sigma'^2=1-\sigma^2$. This solution form has the advantage that the elliptic modulus σ of the Jacobian elliptic function cd is nonnegative and lies between zero and one: $0<\sigma^2<1$.

Notice that the classical solution $z=A\ \mathrm{cn}(u,k^2)$ for the quartic double barrier has the same form as the solution used in Sec. II (and in Ref. [11]) for the quartic well. The *only* difference is the range of values of the elliptic modulus: $k^2<0$ for the quartic double barrier and $0\le k^2\le\frac{1}{2}$ for the quartic well. But the range of the modulus is *not* involved in the manipulations and decompositions made in Ref. [11] (summarized in Sec. II), so that the formulas (2.31)–(2.33) are also valid for the quartic double barrier. Therefore, to evaluate the terms T_i of the GBS rule for the quartic double barrier, it is only necessary to evaluate the *same* three integrals as in Sec. II for the quartic well:

$$\hat{B}_0=\hat{I}_0=\int du\ , \tag{3.5}$$

$$\hat{B}_2=\int_{\Gamma'}\frac{du}{\mathrm{sn}^2(u,k^2)}=\int_{\Gamma'}\frac{du}{\sigma'^2\mathrm{sd}^2(u/\sigma',\sigma^2)}\ , \tag{3.6}$$

$$\hat{I}_2=\int_{\Gamma'}\frac{du}{\mathrm{dn}^2(u,k^2)}=\int_{\Gamma'}\frac{du}{\mathrm{nd}^2(u/\sigma',\sigma^2)}\ . \tag{3.7}$$

The rightmost expression in Eqs. (3.6) and (3.7) has been obtained by using the transformation properties of the Jacobian elliptic functions with respect to the negative modulus [20]. Here Γ' is the integration path depicted in Fig. 1(b), i.e., the path corresponding [through Eq. (3.3)] to the integration path Γ in the complex u plane. This runs from u_0 to $u_0+4\hat{K}$, where $\hat{K}=\sigma'K(\sigma^2)$, since the period of the classical solution (3.3) in u is $4\sigma'K(\sigma^2)$.

Evaluation [21] of the integrals of Eqs. (3.5)–(3.7) shows that $\hat{B}_0$, $\hat{I}_0$, $\hat{B}_2$, and $\hat{I}_2$ are also given by Eqs. (2.37)–(2.39), but now $(k^2<0)$

$$\hat{K}\equiv\sigma'K(\sigma^2)\ , \tag{3.8}$$

$$\hat{E}\equiv\frac{E(\sigma^2)}{\sigma'}\ . \tag{3.9}$$

Notice that $\hat{B}_0$, $\hat{I}_0$, $\hat{B}_2$, and $\hat{I}_2$ have, in terms of $\hat{K}$ and $\hat{E}$, the same values as for the quartic well. Therefore terms T_i of the GBS quantization condition for quartic double barriers have the same form as for quartic wells, Eqs. (2.42) and (2.43), but now with $\hat{K}$ and $\hat{E}$ given by (3.8) and (3.9).

The first five resonance energies of five quartic double barriers calculated by using the GBS rule together with the values obtained by Drummond [3] by means of a perturbation method are listed in Tables I and II. The numerical energy values that appear in the present paper are in units with $\hbar^2/2\mu=1$. The results for the first potential (Table I) are very good: the first five energies calculated with the two-term approximation are accurate to five or six significant figures, and they are accurate to nine or ten figures if the five-term GBS rule is used. For the other four quartic double barriers (Table II) the number of terms n (from two to five) employed in the GBS rule to evaluate each energy is the highest that verifies $|T_n|<|T_{n-1}|<\cdots$. [Remember that the series expansion (3.2) is, in general, only asymptotic and in some cases

TABLE I. The first five energy levels (in units with $\hbar^2/2\mu=1$) of the quartic double barrier $V(z)=z^2-0.01z^4$ calculated by means of the GBS rule of one to five terms. The results are compared with those obtained by Drummond in Ref. [3] using a perturbation technique. The GBS rule of one term is the usual Bohr-Sommerfeld rule.

	E_0	E_1	E_2	E_3	E_4
Ref. [3]	0.992 363 220 6	2.961 401 903 5	4.898 302 036 6	6.801 432 758 5	8.668 928 127 8
BS	0.996 223 064 6	2.965 501 459 6	4.902 678 065 6	6.806 131 467 3	8.674 008 758 4
GBS-2	0.992 365 422 6	2.961 404 737 8	4.898 305 751 9	6.801 437 733 5	8.668 934 960 3
GBS-3	0.992 363 227 5	2.961 401 914 4	4.898 302 054 5	6.801 432 788 7	8.668 928 181 1
GBS-4	0.992 363 220 7	2.961 401 903 6	4.898 302 036 8	6.801 432 758 9	8.668 928 128 9
GBS-5	0.992 363 220 6	2.961 401 903 5	4.898 302 036 6	6.801 432 758 5	8.668 928 127 9

TABLE II. The first five energy levels of four quartic double barriers $V(z)=z^2-\lambda z^4$, where λ is 0.02, 0.03, 0.04, and 0.05, obtained from the GBS rule (a) and from Ref. [3] (b). An asterisk beside a value means that it has been calculated using the GBS rule of four terms; a dagger, the same but with three terms; a double dagger, the same but with two terms. No symbol means that the value has been calculated with the GBS rule of five terms.

		$z^2-0.02z^4$	$z^2-0.03z^4$	$z^2-0.04z^4$	$z^2-0.05z^4$
E_0	(a)	0.984 427 670 4	0.976 146 247 1	0.967 452 516	0.958 243 06
	(b)	0.984 427 669 8	0.976 146 197 4	0.967 451 234	0.958 233 36
E_1	(a)	2.920 282 165 9	2.875 949 52	2.827 110	2.770 93*
	(b)	2.920 282 161 3	2.875 948 30	2.827 103	2.771 26
E_2	(a)	4.786 335 09	4.659 258	4.501 0*	4.254†
	(b)	4.786 335 05	4.659 247	4.504 5	4.315
E_3	(a)	6.573 553	6.290 05*	5.86‡	
	(b)	6.573 552	6.291 08	5.91	
E_4	(a)	8.268 8	7.629†		
	(b)	8.268 8	7.698		

it is possible that $|T_j|>|T_{j-1}|$.] The agreement is good, although it worsens as the energies approach the top of the potential, as is expected since the term Δ neglected in (3.2) increases as the energies approach the top of the potential.

IV. GBS RULE FOR THE QUARTIC DOUBLE WELL ($v_2<0, v_4>0$)

For symmetric double wells, the phase-integral quantization condition of order $2N+1$ is the same as the phase-integral quantization condition of order $2N+1$ of the single well, Eq. (2.13), but now [23]

$$\Delta=\frac{\pi}{4}-\tfrac{1}{2}\arg[F_{12}(-x_4,x_4)]+\arg[F_{11}(x_4,+\infty)] \pm\tfrac{1}{2}\arctan[\exp(-K)/F_{22}(-x_4,x_4)]\ , \qquad (4.1)$$

where x_4 is a point of the real axis on the classically allowed region, K is a real quantity, and F_{11}, F_{12}, and F_{22} are elements of the F matrix defined in [18]. Neglecting the Δ term, one finds the generalized Bohr-Sommerfeld rule of $N+1$ terms (order $2N+1$) for symmetric double wells:

$$\sum_{i=0}^{N} T_i=\tfrac{1}{2}\int_\Gamma \sum_{i=0}^{N} Z_{2i}dz=(n+\tfrac{1}{2})\pi\ ,\quad n=0,1,2,\ldots\ , \qquad (4.2)$$

where use has been made of the equality between Eqs. (2.15) and (2.16).

For the quartic double well, the general solution of the classical equation of motion, Eq. (2.20), is

$$z=A\ \mathrm{cn}(u,k^2)=A\ \mathrm{dn}(u/\eta,\eta^2)\ , \qquad (4.3)$$

where u, γ, and k^2 are given by Eqs. (2.26), (2.27), and (2.28), respectively. I will distinguish two possibilities: (i) $\mathcal{E}>0$, and hence $\frac{1}{2}\le k^2<1$, and (ii) $\mathcal{E}<0$, and hence $k^2>1$.

When the energy is greater than zero, the analysis made in Sec. II for the quartic well can be strictly reproduced for the quartic double well, the *only* difference being that now $\frac{1}{2}\le k^2\le 1$ instead of $0\le k^2\le\frac{1}{2}$ for the quartic well. Therefore the terms T_i are the same as those given in Sec. II for the quartic well, Eqs. (2.42) and (2.43).

When $\mathcal{E}<0$, one has $k^2>1$ and, using the transformation properties of the Jacobian elliptic functions with respect to reciprocal modulus [20], the classical solution can be expressed in terms of the Jacobi elliptic function dn [see Eq. (4.3)] with an elliptic modulus η given by

$$\eta=\frac{1}{k}\ . \qquad (4.4)$$

This way of expressing the solution has the advantage that the elliptic modulus lies between zero and one: $0\le\eta^2<1$. The GBS quantization rule is given by Eq. (4.2), the integration path Γ being a closed loop in the complex z plane encircling the two classical turning

TABLE III. Energy levels for the quartic double well $V(z)=-50z^2+z^4$ calculated by means of the GBS quantization condition of n terms (GBS-n) with $n=1,2,3,4,5$. These values are compared with those given by Hodgson and Varshni in Ref. [13].

	E_0	E_{20}	E_{38}
Ref. [13]	−615.020 090 902 757 816 6	−422.068 788 468 852	−261.112 800 996 988 49
BS	−615.015 042 736 058 959 3	−422.062 444 427 153	−261.104 188 796 919 32
GBS-2	−615.020 090 788 697 919 6	−422.068 788 160 819	−261.112 799 864 301 05
GBS-3	−615.020 090 902 744 364 8	−422.068 788 468 806	−261.112 800 996 094 02
GBS-4	−615.020 090 902 757 812 8	−422.068 788 468 891	−261.112 800 996 986 72
GBS-5	−615.020 090 902 757 816 6	−422.068 788 468 891	−261.112 800 996 988 48

TABLE IV. Energy levels for the quartic double well $V(z)=-5z^2+z^4$ calculated by means of the GBS quantization condition of n terms (GBS-n) with $n=1,2,3,4,5$. These values are compared with those given by Hodgson and Varshni in Ref. [13].

	E_0	E_1	E_{20}	E_{38}	E_{39}
Ref. [13]	−3.410 142 76	−3.250 675 36	96.101 737 842 7	244.366 964 364 5	253.583 300 287 527
BS	−3.254 226 49	−3.254 226 49	96.091 968 480 0	244.360 394 139 1	253.576 836 926 231
GBS-2	−3.327 726 86	−3.327 726 86	96.101 741 071 9	244.366 964 908 9	253.583 300 793 432
GBS-3	−3.333 438 34	−3.333 438 34	96.101 737 839 5	244.366 964 364 6	253.583 300 287 561
GBS-4	−3.336 063 71	−3.336 063 71	96.101 737 842 7	244.366 964 364 6	253.583 300 287 526
GBS-5	−3.339 078 73	−3.339 078 73	96.101 737 842 7	244.366 964 364 6	253.583 300 287 526

points of a single well but no other zeros or singularities of $Q^2(z)$ [see Fig. 1(a)]. The classical solution $z=A\,\mathrm{cn}(u,k^2)$ of the double well has the same form as the classical solution used in Sec. II for the quartic well. Therefore formulas (2.31)–(2.33) are also valid for the double well. The difference is that now $k^2>1$ and the integration path Γ' in the complex u plane, corresponding to Γ in the complex z plane [through Eq. (4.3)], goes from u_0 to $u_0+4\hat{K}$, where $\hat{K}=\frac{1}{2}\eta K(\eta^2)$, since the period in u of the classical solution, Eq. (4.3), is $2\eta K(\eta^2)$.

Repeating for this case the analysis made by Lakshmanan, Karlsson, and Fröman [11] for the quartic well will again convince us that it is only necessary to know

$$\hat{B}_0=\hat{I}_0=\int_{\Gamma'}du\ , \tag{4.5}$$

$$\hat{B}_2=\int_{\Gamma'}\frac{du}{\mathrm{sn}^2(u,k^2)}=\int_{\Gamma'}\frac{du}{\eta\,\mathrm{sn}^2(u/\eta,\eta^2)}\ , \tag{4.6}$$

$$\hat{I}_2=\int_{\Gamma'}\frac{du}{\mathrm{dn}^2(u,k^2)}=\int_{\Gamma'}\frac{du}{\mathrm{cn}^2(u/\eta,\eta^2)}\ , \tag{4.7}$$

to evaluate the terms T_i. The rightmost expression of Eqs. (4.6) and (4.7) has been obtained using the transformation properties of Jacobian elliptic functions with respect to the reciprocal modulus (Jacobi's real transformation) [20]. Solving these rightmost integrals [21], it is found that $\hat{B}_0$, $\hat{I}_0$, $\hat{B}_2$, and $\hat{I}_2$ are also given by Eqs. (2.37)–(2.39), but now $(k^2>1)$

$$\hat{K}\equiv\tfrac{1}{2}\eta K(\eta^2)\ , \tag{4.8}$$

$$\hat{E}\equiv\tfrac{1}{2}\eta[E(\eta^2)-\eta'^2K(\eta^2)]\ . \tag{4.9}$$

Notice that $\hat{B}_0$, $\hat{I}_0$, $\hat{B}_2$, and $\hat{I}_2$ have, in terms of $\hat{K}$ and $\hat{E}$, the same expressions as in the quartic well case. Therefore, for quartic double wells with energies $\mathcal{E}<0$, the terms T_i of the GBS quantization condition have the same form as for quartic wells, Eqs. (2.42) and (2.43), but now with $\hat{K}$ and $\hat{E}$ given by Eqs. (4.8) and (4.9).

Tables III and IV list the values of the energy levels of two quartic double wells taken from Ref. [13], and the unsplit energy levels obtained by means of the GBS rule with from one to five terms. The GBS results for the potential $V(z)=-50z^2+z^4$, Table III, are, in general, very good. For example, the energy of the ground state agrees to nine significant figures using the GBS rule of only two terms and to 19 significant figures using five terms. For the double well $V(z)=-5z^2+z^4$, Table IV, the barrier between the wells is thin enough to make the energy splitting large. In these cases, the GBS rule only serves to evaluate roughly the mean of the two split energies. The mean value of E_0 and E_1 taken from Ref. [13] is $-3.330\,409\,06$, which is close to the values obtained using the GBS rule. The results are once again good. For instance, the GBS rule of four terms gives the energy of the 39th excited state to 14 significant figures.

V. CONCLUDING REMARKS

We have found the first five terms of the generalized Bohr-Sommerfeld rule (Bohr-Sommerfeld quantization condition of higher order) for quartic double barriers and for quartic double wells. In terms of the functions $\hat{K}$ and $\hat{E}$, this generalized rule has the same form [see Eqs. (2.15) and (2.42)] for *all* three quartic oscillators (well, double barrier, and double well). These functions are defined by (i) Eqs. (2.40) and (2.41) when $0\le k^2\le\frac{1}{2}$ (quartic well) [8,9] *and* also when $\frac{1}{2}\le k^2<1$ (quartic double well with $\mathcal{E}>0$); (ii) Eqs. (3.8) and (3.9) when $k^2<0$ (quartic double barrier, $\mathcal{E}<V_{\max}$); and (iii) Eqs. (4.8) and (4.9) when $k^2>1$ (quartic double well with $\mathcal{E}<0$).

These expressions have been used to calculate resonance energies of quartic double barriers, and energy levels of quartic double wells, by means of the Bohr-Sommerfeld quantization condition of n terms, with $n=1,2,\ldots,5$ (first- to ninth-order GBS rule). Comparison with published values shows that in some cases the agreement is excellent (see Tables I–IV).

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