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COMMENTS ON THE METHOD OF HARMONIC BALANCE IN WHICH JACOBI ELLIPTIC FUNCTIONS ARE USED

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A harmonic balance method is presented in which Jacobi elliptic functions are used in the trial solution instead of circular functions to obtain approximate periodic solutions of the oscillator $\ddot{x}+F(x,\dot{x})=0$. Conditions for the method to work well are the usual ones of the current method of harmonic balance, and that $x(t)$ must pass through zero. The procedure for obtaining a higher order approximation is described, and in particular two criteria for chosing the elliptic function parameter m are discussed. Illustrative examples are presented with F being diverse polynomials of x.

1. INTRODUCTION

The current method of harmonic balance [1-3] is a very simple method for obtaining approximate periodic analytical solutions of non-linear oscillators and, unlike perturbation methods, can give good results even for strongly non-linear oscillators. If one uses this method in its first approximation, these good results appear "as long as the behaviour of the motion is close to harmonic" [4]. The reason is because in the first approximation of this method the unknown exact solution $x(t)$ is approximated (as in every method of linearization) by a cosenoidal trial solution $\tilde{x}(t)=A\cos(\omega t)$. The fitting coefficients A and ω are chosen by a different criterion according to each linearization method. In the method of harmonic balance the criterion is based on the principle of harmonic balance: that the approximate solution $\tilde{x}$ must satisfy the equation of the non-linear oscillator

$$\ddot{x}+F(x,\dot{x})=0 \tag{1.1}$$

in its first harmonics: that is, to assume that the largest harmonic of $\ddot{\tilde{x}}$ must be equal to the largest harmonic of $-F(\tilde{x},\dot{\tilde{x}})$. Then, in the first approximation, the method consists of seeking a constant, A, and a function, $\varphi(t)=\omega t$, that make $\tilde{x}=A\cos\varphi$ satisfy equation (1.1) in the first, largest harmonic. But if $x(t)$ is not close to a cosenoidal function there is no choice of A and ω that fits $\tilde{x}(t)$ to $x(t)$: in other words, $\tilde{x}(t)=A\cos(\omega t)$ is a bad trial solution.

It is clear that the fit will probably be better if one uses a trial solution of a more general class of oscillating functions that include circular functions as a particular case. The author and collaborators have applied this idea in a series of papers [5-10] in which $A\,\mathrm{cn}(\omega t, m)=A\cos(\varphi)$, with $\varphi=\mathrm{am}(\omega t, m)=\mathrm{am}(\psi, m)$ is the "more general class of oscillating functions" (the notation of reference [11] is used here for the elliptic functions). Note that when $m=0$ these functions reduce to circular functions. In references [5, 6] the idea was applied to a method of Krylov-Bogoliubov type following the work of Barkham and Soudack [12-15], Christopher [16], and Christopher and Brocklehurst [17]. In reference [7] it was applied to a Galerkin method and in reference [8] to a Rayleigh

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method. Finally, in references [9, 10] an elliptic harmonic balance method (the EHB method) was presented in which Jacobi elliptic functions as trial solutions. The EHB method and the current harmonic balance method that uses circular functions (the CHB method) are very similar: in both the principle of harmonic balance is used to seek constants A_n, B_n, and a function, $\varphi(t)$, so that $\tilde{x} = \sum A_n \cos(n\varphi) + \sum B_n \sin(n\varphi)$ satisfies the non-linear oscillator equation (1.1), at least in its largest harmonics. The difference between the two methods is that $\varphi(t) = \omega t$ is a linear function that depends only on the ω parameter in the CHB method, whereas in the EHB method $\varphi(t) = \mathrm{am}(\omega t, m)$ is a non-linear function (see Figure 1) that depends on two parameters, ω and m. This function is linear only when $m = 0$ and both methods agree. In what follows the set of functions $\{\cos(n\varphi), \sin(n\varphi)\}$ in which $\varphi = \mathrm{am}(\omega t, m)$ is called the set of elliptic harmonics (see Appendix 1 for more details).

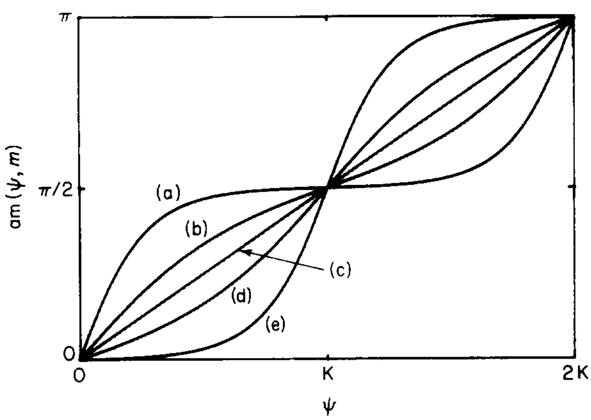

Figure 1. The am (ψ, m) function for (a) $m = 0{\cdot}9999$, (b) $m = 0{\cdot}9$, (c) $m = 0$, (d) $m = -9$ and (e) $m = -999$, with ψ in the interval [0, 2K]. The notation is $K = K(m)$. This interval is enough to show the function $\mathrm{am}(\psi, m)$ since $\mathrm{am}(\psi + 2K, m) = \pi + \mathrm{am}(\psi, m)$.

The aim of this paper is to do the same for the EHB method as Mickens did for the CHB in reference [1]. The present paper is kept as similar as possible to Micken's paper so that the EHB method will be easier to understand, and to compare and contrast. In section 2 the advantages and the applicability conditions of the EHB method are given, and a systematic procedure for obtaining the EHB approximate solution is presented. In section 3 the procedure for higher order approximation in the EHB method is presented, and two criteria for choosing the optimal elliptic parameter m of the approximate solution are given. In section 4 the method is illustrated with some examples. Conclusions are summarized in section 5.

2. THE EHB METHOD

As the bases of the EHB and CHB method are the same, i.e., to take $\tilde{x} = \sum A_n \cos(n\varphi) + \sum B_n \sin(n\varphi)$ as the approximate solution and obtain A_n, B_n, and φ by using the principle of harmonic balance, their principal advantages are also the same (see reference [1]): (i) equation (1.1) can be strongly non-linear; (ii) the limit cycles and their features are easily obtained [9, 10]; (iii) the method is straightforward and, when it works, it gives good approximate solutions.

Also the EHB method works well if certain conditions are satisfied: (a) $F(x, \dot{x})$ is a finite sum of terms of the form $x^i \dot{x}^j$, where i and j are non-negative integers and $i + j$ is odd, so that $F(-x, \dot{x}) = -F(x, \dot{x})$; (b) the oscillating solution $x(t)$ passes through $x=0$; (c) the elliptic harmonics of $x(t)$ of order higher than those that appear in the trial solution $\tilde{x}(t)$ are small. The justification of conditions (a) and (c) is the same as for the CHB method; the justification of condition (b) is based on the nature of the Jacobi elliptic functions.

Condition (a) is based on experience [1]: when this condition is not satisfied it is not possible to assure that the CHB method will work well (for more details about these difficulties, see section 2.3.4 of reference [18]). It is then natural to assume that these restrictions must also be applicable to the EHB method, since the CHB method is only a particular case.

Condition (b) prevents the use of $\tilde{x} = A\cos\varphi^* = A\,\mathrm{dn}\,(m^{1/2}\omega t, 1/m) \equiv A\,\mathrm{dn}$ with $\varphi^* = \mathrm{am}(\omega t, m>1)$ as trial solution, since in this class of trial solution $\tilde{x}(t)$ does not cross $x=0$. The use of $A\,\mathrm{dn}$ as trial solution is not allowed because—at the very least—$\varphi^*(t)$ does not take values inside the interval $(\pi/2, 3\pi/2)$, because in this interval $\cos\varphi^*$ is negative and dn is *always* positive. Thus $\{\cos(n\varphi^*), \sin(n\varphi^*)\}$ is not an orthogonal set. This prevents $x(t)$ being expanded in a Fourier series in terms of $\cos(n\varphi^*)$ and $\sin(n\varphi^*)$. But the possibility of this expansion is a fundamental assumption in the methods of harmonic balance. Therefore the EHB method cannot be applied (as was done incorrectly in reference [10]) to oscillators that do not satisfy condition (b).

Condition (c) comes from the harmonic balance assumption that the largest harmonics of the Fourier expansion of $x(t)$ appear in the trial solution $\tilde{x}(t)$.

One can now look at the procedure for obtaining an approximate analytical solution of (1.1), in sequence.

(I) One checks that conditions (a) and (b) are satisfied.

(II) One takes

$$\tilde{x} = A\cos\varphi \tag{2.1}$$

with $\varphi = \mathrm{am}(\psi = \omega t, m)$ as the *solution to first approximation* of equation (1.1), where the time origin is chosen in such a way that $\dot{x}(t=0)=0$. Substituting expression (2.1) into equation (1.1), using

$$(\mathrm{d}^2\cos\varphi/\mathrm{d}\psi^2) = -(1-m/2)\cos\varphi - \tfrac{1}{2}m\cos 3\varphi \tag{2.2}$$

and expanding the resulting expression in a Fourier series, one finds

$$\sum h_n\cos(n\varphi) + \sum g_n\sin(n\varphi) = 0.$$

(III) One assumes that expression (2.1) satisfies equation (1.1) at least in its largest harmonics, i.e., in its first harmonics.

(III-A) For conservative oscillators (in which $g_n = 0$) one sets $h_1(\omega, A, m, \alpha) = 0$. Hence one obtains $\omega = \omega(A, m, \alpha)$, where α collectively denotes any parameter which appears in the non-linear function $F(x, \dot{x})$.

(III-B) For non-conservative oscillators one sets $h_1(\omega, A, m, \alpha) = 0$ and $g_1(\omega, A, m, \alpha) = 0$. By solving these equations one obtains the parameters that define the possible limit cycles: $\omega_i = \omega_i(m, \alpha)$, $A_i = A_i(m, \alpha)$.

(IV) For non-conservative oscillators one evaluates the stability of the limit cycles.

(V) One checks condition (c) that the higher harmonics of the solution are negligible. In practice, this means that if one takes

$$\tilde{x}(t) = A\cos\varphi + B\cos 3\varphi \tag{2.3}$$

as the trial *solution to a second approximation* and again carries out steps (I), (II) and (III) one must find that $|B/A| \equiv |y| \ll 1$. With this trial solution, in step (III-A) one sets

$h_3=0$ as well as $h_1=0$ to obtain the value of the new parameter B. Similarly, in step (III-B) one sets $h_3=0$ as well as $h_1=0$ and $g_1=0$ to obtain B. A trial solution in the second approximation of the form (2.3) has been assumed because, under condition (a) and with the origin of time chosen suitably so that $\dot{x}(t=0)=0$, only odd cosine harmonics can appear in the solution $x(t)$. In the next section it will be shown how to carry out this step (V).

Notice that the approximate solution that one finds following the above procedure is not yet completely determined, since ω, for conservative oscillators, or ω_i and A_i for non-conservative oscillators, are functions of m and one has not yet seen how to fix its value. The systematic choice of $m=0$ leads to the CHB method, but, except for $m>1$ due to condition (b), any other value should be valid, and better in some cases. However, as will be seen in the next section, it is possible to formulate procedures for choosing the value of m that leads to the best (according to some criterion) approximate solution $\tilde{x}(t)$.

3. HIGHER ORDER APPROXIMATIONS IN THE EHB METHOD

For the sake of simplicity, and without loss of generality, it is shown here how to carry out step (V) for conservative oscillators only: i.e., for oscillators with the equation form

$$\ddot{x}+F(x)=0. \tag{3.1}$$

Substituting

$$\tilde{x}=A(\cos\varphi+y\cos 3\varphi) \tag{3.2}$$

into equation (3.1) (step II), using equation (2.2) and the relation (see Appendix 2)

$$d^2\cos(3\varphi)/d^2\psi=-\tfrac{3}{2}m\cos\varphi-9(1-m/2)\cos(3\varphi)-3m\cos(5\varphi), \tag{3.3}$$

where $\psi=\omega t$, one obtains

$$\ddot{\tilde{x}}+F(\tilde{x})=\sum h_n\cos(n\varphi)=0,$$

with

$$h_1=-a_1A\omega^2+a_2,\qquad h_3=-b_1A\omega^2+b_2,\qquad h_5=-c_1A\omega^2+c_2, \tag{3.4a-c}$$

where

$$a_1=a_1(y,m)=(1-m/2)-3my/2,\qquad b_1=b_1(y,m)=9y+\tfrac{1}{2}m(1-9y),$$
$$c_1=c_1(y,m)=3my$$

and

$$F(\tilde{x}=A\cos\varphi+Ay\cos 3\varphi)=a_2\cos\varphi+b_2\cos 3\varphi+c_2\cos 5\varphi+\cdots.$$

Notice that this procedure is the same as the procedure in the CHB method. Therefore the expressions for a_2, b_2, and c_2 are the same as those obtained in the CHB method. The difference is that the harmonics a_1, b_1, c_1 of $\ddot{\tilde{x}}$ are not the same since they depend on the parameter m.

Next (step III) the values of $\omega=\omega(A,m,\alpha)$ and $y=y(A,m,\alpha)$ are obtained from the equations $h_1=0$ and $h_3=0$. By eliminating $A\omega^2$ from $h_1=h_3=0$, equations (3.4a, b), one finds

$$\tfrac{1}{2}m(a_2+b_2)+9ya_2-\tfrac{9}{2}mya_2-b_2-\tfrac{3}{2}myb_2=0. \tag{3.5}$$

Expressing a_2, b_2, and c_2 in powers of y,

$$a_2=\sum \alpha_i(A)y^i, \qquad b_2=\sum \beta_i(A)y^i, \qquad c_2=\sum \gamma_i(A)y^i,$$

and substituting these expressions into equation (3.5) one obtains

$$\sum \otimes_n y^n=0, \tag{3.6}$$

with

$$\otimes_n=[\tfrac{1}{2}(\alpha_n+\beta_n)-\tfrac{9}{2}\alpha_{n-1}-\tfrac{3}{2}\beta_{n-1}]m+9\alpha_{n-1}-\beta_n, \tag{3.7}$$

where $\alpha_i=\beta_i=\gamma_i=0$ if $i<0$. The value of y is obtained from equation (3.6). So, to first order in y, with powers higher than unity neglected, one has

$$y(A, m, \alpha)\equiv y_1=-\otimes_0/\otimes_1.$$

To second order in y, with powers higher than two neglected, one finds

$$y(A, m, \alpha)\equiv y_2=[-\otimes_1\pm(\otimes_1^2-4\otimes_2\otimes_0)^{1/2}]/2\otimes_2.$$

The value of y_2 chosen is that of smallest modulus. Substituting this value into $h_1=0$, one finds the ω value for the second approximation of the ECB method using terms to second order in y: $\omega^2=a_2/Aa_1$. If one sets $y=0$ in this last equation, i.e., if one works only with the first approximation of the ECB method, one finds

$$\omega^2=\alpha_0/[A(1-m/2)]. \tag{3.8}$$

The simplicity (or complexity) of the CHB and the EHB method is the *same*: deductions and expressions calculated up to now are the same in both methods, the only difference is that m is *always* zero in the CHB method. Of course, the question for the EHB method is: what value of m to use? Here two criteria to fix m are suggested and discussed (of course other criteria can be formulated).

Criterion I. The parameter m is chosen so that $\tilde{x}=A[\cos(\varphi)+y\cos(3\varphi)]$ satisfies equation (1.1) at least in its first two harmonics ($h_1=h_3=0$) with $|y|$ as small as possible, i.e., with $y=0$. Notice that this criterion can equivalently be formulated thus: m is such that the solution to first approximation, $\tilde{x}=A\cos\varphi$, is equal to the solution to second approximation, $\tilde{x}=A\cos(\varphi)+Ay\cos(3\varphi)$. If $y=0$, from equation (3.6) one finds $\otimes_0=0$, and then, from equation (3.7),

$$m=2b_2/(a_2+b_2), \tag{3.9}$$

and from equation (3.8),

$$\omega^2=(a_2+b_2)/A. \tag{3.10}$$

The fact that $a_2=\alpha_0$ and $b_2=\beta_0$ when $y=0$ has been used. This criterion was used in reference [10] although it was not recognized there in this form.

Criterion II. The parameter m is chosen so that $\tilde{x}=A\cos(\varphi)+Ay\cos(3\varphi)$ satisfies equation (3.1) in the largest number of harmonics: $h_{2n+1}=0$ with $n=0, 1, \ldots$. In practice this means that one searches for values of m, y and ω that make $h_1=h_3=0$, and $|h_5|$ as small as possible (ideally $h_5=0$), but, of course, under the restrictions that $m<1$ and $|y|\ll 1$.

4. ILLUSTRATIVE EXAMPLES

The first two examples,

$$\ddot{x}+x=0 \quad \text{and} \quad \ddot{x}+x^3=0, \tag{4.1, 4.2}$$

are particular cases of the third example oscillator

$$\ddot{x}+c_1x+c_3x^3=0. \tag{4.3}$$

By using criteria I and II the EBH method will be shown to give the exact analytical solution for the above three oscillators. The other examples are a hard oscillator

$$\ddot{x}+x^5=0, \tag{4.4}$$

a soft-hard oscillator

$$\ddot{x}-x^3+x^5=0, \tag{4.5}$$

and a hard-soft oscillator

$$\ddot{x}+x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}=0. \tag{4.6}$$

From these examples some conclusions will be reached: (i) that criterion I is simple and accurate; (ii) that criterion II is less straightforward than criterion I but that it leads to the smallest higher harmonics, i.e., it gives more accurate results; and (iii) that criterion I is preferable because, although it is a little less accurate than criterion II, it is much simpler.

The results are given in Tables 1-3. In the first columns appear the elliptic parameter m, the frequency ω, and the coefficient $y=B/A$ of the approximate solution (3.2) obtained by using the procedures expounded in section 3. Also given is the value of the fifth harmonic of equation (3.1) as obtained by using expression (3.2) as solution, i.e., h_5 is given (h_1 and h_3 are zero): this value gives an estimate of the precision of the approximate solution (3.2) since, usually, this approximate solution is better when h_5 is smaller. The results are obtained with y used to first order (with powers of y higher than y neglected) although y is used to second order if the results are significantly better. The last three columns give the ratios between the coefficients of the first four harmonics, A_3/A_1, A_5/A_1 and A_7/A_1, of the solution $x(t)$ evaluated numerically; the procedure for obtaining these coefficients is expounded in Appendix 1. The ratio $y_N\equiv A_3/A_1$ must be compared with y. The ratios A_5/A_1 and A_7/A_1 serve to check the harmonic balance assumption (see condition (c)) that the higher order harmonics are negligible with respect to the first.

Oscillator (4.1). One has $F(A\cos\varphi+Ay\cos 3\varphi)=A\cos\varphi+Ay\cos 3\varphi$, i.e., $\alpha_0=\beta_1=A$ and the other coefficients are zero. Criteria I and II give $y=0$, $m=0$, $\omega^2=1$, *i.e.*, they give the solution $x=A\cos t$, the *exact solution*. The results for the choice $m=1/2$ are presented in Table 1.

TABLE 1

Results obtained by using the EHB method for the oscillators $\ddot{x}+x^{n+1}=0$ with $n=0,2,4$; column C gives the criterion that was used to obtain the parameter m: I, criterion I; II, criterion II; blank, no criterion (m is given directly); the 2 added in this column means that calculations include terms to second order in y, i.e., $y=y_2$; the notation 8·7(2) means $8\cdot 7\times 10^{-2}$

Equation	C	m	ω^2/A^n	h_5	y	$y_N=A_3/A_1$	A_5/A_1	A_7/A_1
$\ddot{x}+x=0$, $n=0$		0·5	1·3913	8·7 (2)	−0·0417	−0·0430	−3·7 (3)	4·0 (4)
$\ddot{x}+x^3=0$, $n=2$		0	0·7857	3·6 (2)	0·0476	0·0451	1·9 (3)	8·4 (5)
$\ddot{x}+x^5=0$, $n=4$	I	0·667	0·9375	6·2 (2)	0	−0·0025	4·9 (3)	−6·7 (4)
	II	0·580	0·8962	5·8 (2)	0·0142	0·0122	4·3 (3)	−2·6 (4)
	2	0	0·7490	1·5 (1)	0·0682	0·0684	9·5 (3)	1·3 (3)

Oscillator (4.2). Now $\alpha'_0=\alpha'_1=\gamma'_1=\gamma'_2=3/4$, $\alpha'_2=\beta'_1=3/2$, $\beta'_0=1/4$, $\beta'_2=\gamma'_2=0$ where $(\ )=(\ )'A^3$. Criteria I and II give $y=0$, $m=1/2$, $\omega^2=A^2$, i.e., they give the *exact solution* $x=A\cos\varphi$ with $\varphi=\mathrm{am}(\omega t, 1/2)$. In Table 1 the results are given for the choice $m=0$, i.e., for the CBH method [1].

Oscillator (4.3). The oscillator coefficients c_1 and c_3 can take any value. Condition (b) requires that $c_3A^2/c_1<-1$ if $c_1<0$ and $c_3>0$. The coefficients α_i, β_i, γ_i are evaluated from the coefficients of the two previous oscillators: thus $\alpha_0=c_1A+\frac{3}{4}c_3A^3$, $\beta_1=c_1A+\frac{3}{2}c_3A^3$, etc. With these coefficients, it is easy to show that criteria I and II give $y=0$, $m=\frac{1}{2}c_3A^2/(c_1+c_3A^2)$ and $\omega^2=c_1+c_3A^2$. That is, the trial solution $\tilde{x}=A\cos\varphi$ with $\varphi=\mathrm{am}(\omega t, m)$ is the *exact solution.* Therefore, the EHB method gives the exact solution for the cubic oscillator (4.3) in the same way that the CHB method gives the exact solution for the linear oscillator $\ddot{x}+c_1x=0$.

Oscillator (4.4). The coefficients α_i, β_i, γ_i of $(A\cos\varphi+Ay\cos 3\varphi)^5$ are $\alpha'_0=10$, $\alpha'_1=25$, $\alpha'_2=\beta'_1=\beta'_2=\gamma'_2=30$, $\beta'_0=5$, $\gamma'_0=1$, $\gamma'_1=20$, where $(\ )=(\ )'A^5/16$. The results are given in Table 1.

Oscillator (4.5). Condition (b) requires that $A^2>A_c^2=3/2$. The results for the amplitude $A=1{\cdot}3$ (close to A_c) and for $A=5$ are given in Table 2. For $A=1{\cdot}3$ the higher harmonics are larger than usual, although criterion II gives smaller harmoncs than criterion I. Notice that with the CHB method ($m=0$), the assumption $|y|\ll 1$ is hard to defend.

Oscillator (4.6). The force $F(x)$ is a power series of $\sin(x)$ up to order seven. The coefficients α_i, β_i, γ_i of $(A\cos\varphi+Ay\cos 3\varphi)^7$ are $\alpha'_0=35$, $\alpha'_1=147$, $\alpha'_2=441$, $\beta'_0=21$,

TABLE 2

Results obtained by using the EHB method for the oscillator $\ddot{x}-x^3+x^5=0$; the key is the same as in Table 1

Amplitude	C	m	ω^2	h_5	y	$y_N=A_3/A_1$	A_5/A_1	A_7/A_1
1·3	I	0·952	0·988	2·3	0	−0·0351	2·8 (2)	−1·1 (2)
	II2	0·889	0·9874	2·3 (1)	0·0154	0·0103	2·2 (2)	−4·3 (3)
	2	0	0·9570	6·2 (1)	0·1078	0·1460	4·4 (2)	1·3 (2)
5·0	I	0·674	560·9	2·0 (4)	0	−0·0027	5·2 (3)	7·2 (4)
	II	0·588	536·6	1·8 (4)	0·0144	0·0122	4·5 (3)	−2·8 (4)
	2	0	449·3	4·9 (4)	0·0693	0·0697	9·9 (3)	1·4 (3)

TABLE 3

Results obtained by using the EHB method for the oscillator $\ddot{x}+x-x^3/3!+x^5/5!-x^7/7!=0$; the key is the same as in Table 1

Amplitude	C	m	ω^2	h_5	y	$y_N=A_3/A_1$	A_5/A_1	A_7/A_1
1·0	I	−0·093	0·8410	5·0 (4)	0	0·0000	2·3 (5)	1·9 (7)
	II	−0·068	0·8510	4·2 (4)	−0·0014	−0·0014	2·0 (5)	1·4 (8)
		0	0·8807	1·1 (3)	−0·0055	−0·0055	5·4 (5)	−6·3 (7)
2·0	I	−0·578	0·4472	1·4 (2)	0	−0·0002	5·0 (4)	2·1 (5)
	II	−0·390	0·4826	1·1 (2)	−0·0071	−0·0075	4·0 (4)	−2·3 (6)
		0	0·5842	3·2 (2)	−0.0256	−0·0274	1·3 (3)	6·9 (5)
3·0	I	−67·69	6·250 (3)	7·9 (2)	0	−0·0527	1·6 (2)	6·5 (3)
	II	−4·32	6·864 (3)	4·6 (2)	−0·0296	−0·0574	4·5 (3)	−7·5 (4)
		0	0·2552	2·2 (1)	−0·0929	−0·1409	2·7 (2)	−6·3 (3)

$\beta_1'=30$, $\beta_2'=315$, $\gamma_0'=7$, $\gamma_1'=20$, $\gamma_2'=252$, where $(\ )=(\ )'A^7/64$. The results are given in Table 3. The poorest results occur for the amplitude $A=3$, close to the critical point where the motion is not oscillatory. As before, criterion II gives the more accurate value of y and leads to the smallest higher harmonics. Criterion I also gives good results.

5. CONCLUSIONS

A method of harmonic balance (the EHB method) that is a generalization of the usual method of harmonic balance (the CHB method) has been expounded in which Jacobi elliptic functions are used in the trial solutions instead of circular functions. The systematic procedure for obtaining the approximate periodic solutions of oscillators with the general form $\ddot{x}+F(x,\dot{x})=0$ has been presented, as well as the conditions under which the method works well.

It is known that, in its first approximation, the results of the CHB method are good for all oscillators "as long as the motion is close to [a circular] harmonic" [4]. The same assertion is true for the EHB method as long as the motion is close to an *elliptic harmonic*.

The possibility of constructing a general procedure for calculating higher order approximate solutions in the framework of the method of harmonic balance with elliptic harmonics was shown.

Finally, the method presented here is "open" in the sense that different criteria for choosing m lead to different approximations: for example, the special choice $m=0$ leads to the well known CBH method. In this paper two criteria were used. One of these, criterion I, leads to very simple and accurate approximate solutions: they have the simplicity of a first order approximate solution and the accuracy of a second order approximation solution.

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APPENDIX 1

In this appendix it will be shown how to find the expansion of the periodic function $x(\psi)$, with period $4\mathrm{K}(m) \equiv 4\mathrm{K}$, in terms of the periodic set (*elliptic harmonics*)

$$\cos_0(\psi, m) \equiv 1, \qquad \cos_n(\psi, m) \equiv \cos(n\varphi), \qquad \sin_n(\psi, m) \equiv \mathrm{sen}(n\varphi), \tag{A1}$$

where $\varphi = \mathrm{am}(\psi, m)$, $m < 1$ and $n = 1, 2, \ldots$. In other words, one looks for the coefficients A_n, B_n (Fourier coefficients) of

$$x(\psi) = (A_0/2) + \sum A_n \cos(n\varphi) + \sum B_n \sin(n\varphi). \tag{A2}$$

These coefficients can be obtained in two ways, as follows.

First procedure. Using the inverse function $\psi = \mathrm{am}^{-1}(\varphi, m)$ one obtains the function $x(\varphi, m)$ from the function $x(\psi)$: $x(\psi) = x(\psi(\varphi, m)) \equiv x(\varphi, m)$. Then the Fourier coefficients A_n, B_n of expression (A2) are given by a simple Fourier (trigonometric) expansion of $x(\varphi, m)$ in terms of $\cos(n\varphi)$ and $\sin(n\varphi)$:

$$A_n(m) = \frac{1}{\pi}\int_0^{2\pi} x(\varphi, m)\cos(n\varphi)\,\mathrm{d}\varphi, \qquad B_n(m) = \frac{1}{\pi}\int_0^{2\pi} x(\varphi, m)\sin(n\varphi)\,\mathrm{d}\varphi. \tag{A3, A4}$$

One can use a similar procedure when one looks for the Chebyshev expansion of a function $x(\psi)$ in terms of

$$\mathrm{T}_0(\psi) = 1, \quad \mathrm{T}_n(\psi) = \cos[n \arccos(\psi)] = \cos(n\varphi), \tag{A5}$$

where $\varphi = \arccos(\psi)$ and $n = 1, 2, \ldots$. Then, the coefficients A_i of the Chebyshev expansion $x(\psi) = (A_0/2) + \sum A_n \cos(n\varphi)$ can be evaluated simply by using equation (A3).

Second procedure. However, although the above procedure is valid, it is well known that this is not the usual way of obtaining the Chebyshev coefficients A_n of a function $x(\psi)$, where it is not necessary to change the function $x(\psi)$ into the function $x(\varphi)$ The current procedure uses a new set of orthogonal functions defined in the ψ variable. This set is that of expression (A5) (the Chebyshev polynomials) for the Chebyshev expansion. In a similar way, the set (A1) is the new expansion orthogonal set used in the present paper to construct the EHB method. One therefore has

$$x(\psi) = (A_0/2) + \sum A_n \cos_n(\psi, m) + \sum B_n \sin_n(\psi, m),$$

where, upon substituting expression (A1) and the formula (A9) that will be given in Appendix 2, into expressions (A3) and (A4), one has

$$A_n \equiv A_n(m) = \frac{1}{\pi}\int_0^{4K} x(\psi)\cos_n(\psi, m)\,\mathrm{dn}(\psi, m)\,\mathrm{d}\psi,$$

$$B_n \equiv B_n(m) = \frac{1}{\pi}\int_0^{4K} x(\psi)\sin_n(\psi, m)\,\mathrm{dn}(\psi, m)\,\mathrm{d}\psi.$$

In other words, one has defined in expression (A1) a new set of functions

$$\{f_i = \cos_i(\psi, m), \quad g_j = \sin_j(\psi, m), \qquad i = j-1 = 0, 1, 2, \ldots\} \tag{A6}$$

(*elliptic harmonics*) that are orthogonal with respect to the weight factor $w(\psi, m) = \mathrm{dn}(\psi, m)$ over the interval $0 \leqslant \psi \leqslant p = 4K$, since

$$\frac{1}{\pi}\int_0^p w(\psi, m) f_i g_j \,\mathrm{d}\psi = 0, \qquad \forall i,j,$$

$$\frac{1}{\pi}\int_0^p w(\psi, m) f_i f_j \,\mathrm{d}\psi = \frac{1}{\pi}\int_0^p w(\psi) g_i g_j \,\mathrm{d}\psi = \delta_{ij}. \tag{A7}$$

The Fourier expansion with this set of functions is therefore made by means of the standard techniques. Notice that this new set of orthogonal functions becomes the usual trigonometric set when $m = 0$.

Finally, it should be noted that the orthogonal set (A6) is complete, and therefore, closed. The demonstration is direct by using the set definition (A6) and the fact that the usual trigonometric set is complete.

APPENDIX 2

In this appendix relation (3.3) will be deduced. By using the chain rule one has

$$\mathrm{d}^2\cos(3\varphi)/\mathrm{d}\psi^2 = -3(\mathrm{d}^2\varphi/\mathrm{d}\psi^2)\sin(3\varphi) - 9(\mathrm{d}\varphi/\mathrm{d}\psi)^2\cos(3\varphi). \tag{A8}$$

But [11],

$$\mathrm{d}\varphi/\mathrm{d}\psi = \mathrm{d}[\mathrm{am}(\psi, m)]/\mathrm{d}\psi = \mathrm{dn}(\psi, m), \tag{A9}$$

and thus

$$\mathrm{d}^2\varphi/\mathrm{d}\psi^2 = -m\,\mathrm{sn}(\psi, m)\,\mathrm{cn}(\psi, m) = -m\sin(\varphi)\cos(\varphi). \tag{A10}$$

Substituting expressions (A9) and (A10) into equation (A8), using $\mathrm{dn}^2 = (1-m^2) + m\cos^2(\varphi)$, and using the trigonometric relations $\sin(\varphi)\cos(\varphi)\sin^3(\varphi) = [\cos(\varphi) - \cos(5\varphi)]/4$ and $\cos^2(\varphi)\cos(3\varphi) = [\cos(\varphi) + 2\cos(3\varphi) + \cos(5\varphi)]/4$, one obtains equation (3.3). The second derivative of other elliptic harmonics can be obtained in a similar way.