Journal of Sound and Vibration (1989) **130**(2), 332-336

A GENERALIZED GALERKIN METHOD FOR CUBIC OSCILLATORS

1. INTRODUCTION

In a recent paper Chen [1] pointed out that the Galerkin method is not as straightforward as usually supposed: the weighting functions that this method uses are not always known *a priori*. Chen gave a generalized Galerkin procedure for oscillators that avoids this problem. This procedure will be briefly expounded in section 2. The purpose of the present communication is to show that this Galerkin method is especially suitable for cubic oscillators whose trial solutions are given in terms of Jacobi elliptic functions, because the choice of weighting functions for this type of solution is difficult.

In section 3, the generalized Galerkin method is shown to be directly applicable to perturbed *pure* cubic oscillators,

$$\ddot{q}+c_3q^3+\varepsilon h(q,\dot{q})=0, \tag{1}$$

with $c_3>0$ and ε a small parameter. In section 4, the procedure for perturbed cubic oscillators,

$$\ddot{q}+c_1q+c_3q^3+\varepsilon h(q,\dot{q})=0, \tag{2}$$

is shown also to be straightforward, although it is necessary to obtain a *further additional* Galerkin condition, besides the additional condition given by Chen [1].

2. THE GENERALIZED GALERKIN METHOD FOR NON-LINEAR OSCILLATORS

In this section, the method is expounded for equations with only one degree of freedom. The method starts from the differential equation of motion (Lagrange equation):

$$E(q)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}-Q=0, \tag{3}$$

where L is the Lagrangian of the system and Q is a non-potential force. If the exact solution $q(t)$ is replaced with the trial solution $\tilde{q}(t)$, $E(\tilde{q})\equiv\tilde{E}$ is no longer zero. According to the d'Alembert principle, $\tilde{E}$ can be thought of as a "residual" force. In Chen's procedure the unknown parameters of the trial solution are chosen to make the average "residual" work over a certain time interval equal to zero:

$$\int_{t_0}^{t_1}\tilde{E}\delta\tilde{q}\,\mathrm{d}t=0. \tag{4}$$

For problems having periodic solutions, one assumes that the trial solution has the form $\tilde{q}=Ap(\omega t)$, where p is a periodic function with period T, i.e., $p(\omega t+T)=p(\omega t)$. Either the period T (or the half-period) is taken as the time interval of integration.

If ω is known (as in the steady response of forced oscillations, for example) $\delta\tilde{q}=p(\omega t)\delta A$ and equation (4) reduces to the ordinary Galerkin condition

$$\int_0^T\tilde{E}p(\omega t)\,\mathrm{d}t=0.$$

But if ω is unknown (as in self-excited oscillations) ω is also subjected to variation and

$$\delta\tilde{q}=p(\omega t)\delta A+Ap_\omega(\omega t)\delta\omega=p(\psi)\delta A+Atp_\psi(\psi)\delta\omega,$$

where $\psi=\omega t$, $p_\omega\equiv\partial p/\partial\omega$ and $p_\psi\equiv\partial p/\partial\psi$. From equation (4) one now obtains

$$\int_0^T\tilde{E}p(\psi)\,\mathrm{d}t=0,\qquad\int_0^T\tilde{E}Atp_\psi(\psi)\,\mathrm{d}t=0. \tag{5a, b}$$

0022-460X/89/080332+05 $03.00/0

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The first identity is the ordinary Galerkin condition, and the second is the additional Galerkin condition given by Chen.

3. PERTURBED PURE CUBIC OSCILLATORS

The trial solution used for the perturbed pure cubic oscillator, described by equation (1), is the solution of the non-perturbed oscillator $\ddot{q}+c_3q^3=0$, that is,

$$\tilde{q}(t)=A\,\mathrm{cn}\,(\omega t, m=1/2)\equiv A\,\mathrm{cn},$$

with $\omega^2=c_3A^2$, and where m is the parameter of the Jacobi elliptic function cn (the elliptic function notation is that of reference [2]). Then

$$\delta\tilde{q}=\mathrm{cn}\,\delta A-At\,\mathrm{sn}\,\mathrm{dn}\,\delta\omega$$

where $\mathrm{sn}\equiv\mathrm{sn}\,(\psi,1/2)$ and $\mathrm{dn}\equiv\mathrm{dn}\,(\psi,1/2)$. The Galerkin conditions are then

$$\int_0^T\tilde{E}\,\mathrm{cn}\,\mathrm{d}t=0,\qquad \int_0^T\tilde{E}At\,\mathrm{sn}\,\mathrm{dn}\,\mathrm{d}t=0,\tag{6a, b}$$

where $T=4\mathrm{K}(1/2)/\omega=4\mathrm{K}/\omega$, with $\mathrm{K}=1{\cdot}85407\ldots$ the complete elliptic integral of the first kind of parameter $m=1/2$. Two examples follow.

3.1. *The Van der Pol pure cubic oscillator*

The equation is

$$\ddot{q}+c_3q^3-\varepsilon(\alpha-\beta q^2)\dot{q}=0,$$

and therefore

$$\tilde{E}=-A\omega^2\,\mathrm{cn}^3+c_3A^3\,\mathrm{cn}^3+\varepsilon A\omega(\alpha-\beta A^2\,\mathrm{cn}^2)\,\mathrm{sn}\,\mathrm{dn}.$$

Carrying out the integrations of equation (6), one easily finds (see reference [3]) from condition (6a) that $\omega^2=c_3A^2$, and from condition (6b) that

$$A^2=(5\alpha/3\beta)/[2(\mathrm{E}/\mathrm{K})-1]=3{\cdot}6474\alpha/\beta,$$

where $\mathrm{E}=\mathrm{E}(1/2)=1{\cdot}35064\ldots$ is the complete elliptic integral of the second kind of parameter $m=1/2$. These results are the same as those obtained in reference [4] by using a harmonic balance method.

3.2. *Duffing's oscillator*

This oscillator is

$$\ddot{q}+\varepsilon q+c_3q^3=0,\tag{7}$$

and therefore

$$\tilde{E}=(-A\omega^2+c_3A^3)\,\mathrm{cn}^3+\varepsilon A\,\mathrm{cn}.$$

As A and ω are connected in this oscillator, equations (6a) and (6b) are not independent. From equation (6a) one obtains [3]

$$\omega^2=c_3A^2+\varepsilon(6\mathrm{E}/\mathrm{K}-3)=c_3A^2+1{\cdot}3708\varepsilon,$$

but from equation (6b) one also obtains [3]

$$\omega^2=c_3A^2+\varepsilon 6(1-\mathrm{E}/\mathrm{K})=c_3A^2+1{\cdot}6292\varepsilon.$$

As Chen discusses in reference [1], this "contradiction" serves as a measure of the approximation of the solution.

The harmonic balance method expounded in reference [4], gives $\omega^2=c_3A^2+(4/3)\varepsilon$.

A weighted mean "cubication" method for conservative oscillators [5] gives $\omega^2=c_3A^2+(s+7/s+5)\varepsilon$, where s is a parameter chosen so that the approximate solution of equation (7) has the same period as the exact solution. Usually the choice $s=0$ gives good results.

4. PERTURBED CUBIC OSCILLATORS

The trial solution used for the perturbed cubic oscillator of equation (2) is the solution of the non-perturbed oscillator

$$\ddot{q} + c_1 q + c_3 q^3 = 0. \tag{8}$$

Before continuing it will be convenient to define some useful quantities: the non-linearity factor, $\nu = c_3 A^2/c_1$; total energy of oscillation, $En = c_1 A^2 + c_3 A^4/2$; and the maximum or minimum potential, $V_m = c_1^2/(2c_3)$.

Equation (8) has three solutions of the form $\tilde{q} = Ap(\omega t, m)$, where p is a Jacobi elliptic function: (i) $\tilde{q} = A\,\mathrm{cn}\,(\omega t, m) \equiv A\,\mathrm{cn}$, with $\omega^2 = c_1(1+\nu)$, $m = \nu/[2(1+\nu)]$, $T = 4\,\mathrm{K}(m)/\omega$ when $c_1 > 0$, $c_3 > 0$ (hard oscillator) or when $c_1 < 0$, $c_3 > 0$ and $En \geqslant 0$ (soft-hard oscillator); (ii) $\tilde{q} = A\,\mathrm{cd}\,(\omega t, m) \equiv A\,\mathrm{cd}$, with $\omega^2 = c_1(1+\nu/2)$, $m = -\nu/(2+\nu)$, $T = 4\mathrm{K}(m)/\omega$ when $c_1 > 0$, $c_3 < 0$ and $0 < En < V_m$ (hard-soft oscillator); (iii) $\tilde{q} = A\,\mathrm{dn}\,(\omega t, m) \equiv A\,\mathrm{dn}$, with $\omega^2 = c_1\nu/2$, $m = 2(1+1/\nu)$, $T = 2\mathrm{K}(m)/\omega$ when $c_1 < 0$, $c_3 > 0$ and $En \leqslant 0$ (soft-hard oscillator).

When, as in the next example, A, ω and m are unknown, not only A and ω, but also m must be subjected to variation in following Chen's procedure: i.e., one has

$$\delta\tilde{q} = p(\psi, m)\delta A + Atp_\psi(\psi, m)\delta\omega + Ap_m(\psi, m)\delta m.$$

There is, therefore, *another* additional Galerkin condition

$$\int_0^T \tilde{E}Ap_m(\psi, m)\,\mathrm{d}t = 0. \tag{9}$$

4.1. *An example: the Van der Pol cubic oscillator*

This oscillator (also called the Van der Pol-Duffing oscillator) is

$$\ddot{q} + c_1 q + c_3 q^3 - \varepsilon(\alpha - \beta q^2)\dot{q} = 0.$$

For the class (i) of oscillators one has [3]

$$\delta\tilde{q} = \mathrm{cn}\,\delta A - At\,\mathrm{sn}\,\mathrm{dn}\,\delta\omega + A\,\mathrm{sn}\,\mathrm{dn}\,[-m_1\psi - \mathrm{E}(\psi) - m\,\mathrm{sn}\,\mathrm{cd}]\delta m/(2mm_1),$$

where $m_1 = 1 - m$ and where $\mathrm{E}(\psi) \equiv \mathrm{E}(\psi, m)$ is the incomplete elliptic integral of the second kind. The three Galerkin conditions, (5a), (5b) and (9), are now

$$\int_0^{4\mathrm{K}} \tilde{E}\,\mathrm{cn}\,\mathrm{d}\psi = 0, \qquad \int_0^{4\mathrm{K}} \tilde{E}(-At\,\mathrm{sn}\,\mathrm{dn})\,\mathrm{d}\psi = 0,$$

$$\int_0^{4\mathrm{K}} \tilde{E}\{A\,\mathrm{sn}\,\mathrm{dn}\,[-m_1\psi - \mathrm{E}(\psi) - m\,\mathrm{sn}\,\mathrm{cd}]/(2mm_1)\}\,\mathrm{d}\psi, \tag{10}$$

where

$$\tilde{E} = A[-\omega^2(1-2m) + c_1]\,\mathrm{cn} + [-2\omega^2 m + c_3 A^2]A\,\mathrm{cn}^3 + \varepsilon(\alpha - \beta A^2\,\mathrm{cn}^2)A\omega\,\mathrm{sn}\,\mathrm{dn}$$

and $\mathrm{K} = \mathrm{K}(m)$. By using various properties of the elliptic functions [3], it is not too difficult to derive the following system of equations from the above three conditions (10):

$$c_1 - \omega^2(1-2m) = 0, \qquad c_3 A^2 - 2\omega^2 m = 0,$$

$$A^2 = \alpha \int_0^{4\mathrm{K}} \mathrm{sn}^2\,\mathrm{dn}^2\,\mathrm{d}\psi/\beta \int_0^{4\mathrm{K}} \mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{dn}^2\,\mathrm{d}\psi, \tag{11}$$

i.e. (see reference [3]),

$$A^2 = (\alpha/5\beta m)\{[(2m-1)\mathrm{E} - m_1\mathrm{K}]/[2(m^2 + m_1)\mathrm{E} + m_1(m-2)\mathrm{K}]\}$$

where $\mathrm{E}=\mathrm{E}(m)$. In a similar form one finds for the class (ii)

$$c_1-\omega^2(1+m)=0, \qquad c_3A^2+2\omega^2 m=0,$$

$$A^2=\alpha\int_0^{4\mathrm{K}}\mathrm{sd}^2\,\mathrm{nd}^2\,\mathrm{d}\psi\Big/\beta\int_0^{4\mathrm{K}}\mathrm{sd}^2\,\mathrm{nd}^2\,\mathrm{cd}^2\,\mathrm{d}\psi, \tag{12}$$

i.e. (see reference [3]),

$$A^2=(\alpha/5\beta m)\{[(1+m)\mathrm{E}-m_1\mathrm{K}]/[2(m^2+m_1)\mathrm{E}+m_1(m-2)\mathrm{K}]\}.$$

Also, for the class (iii),

$$c_1-\omega^2(m-2)=0, \qquad c_3A^2-2\omega^2=0,$$

$$A^2=\alpha\int_0^{2\mathrm{K}}\mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{d}\psi\Big/\beta\int_0^{2\mathrm{K}}\mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{dn}^2\,\mathrm{d}\psi, \tag{13}$$

i.e. (see reference [3]),

$$A^2=(\alpha/5\beta)\{[(2-m)\mathrm{E}-2m_1\mathrm{K}]/[2(m^2+m_1)\mathrm{E}+m_1(m-2)\mathrm{K}]\}.$$

Solving these systems gives the values of A, ω and m for each limit cycle. The results are good: the expressions (11)-(13) are the same as those obtained using the harmonic balance method (see references [6, 7]).

5. CONCLUSIONS

The generalized Galerkin method expounded by Chen has been shown to be applicable when the trial solutions are Jacobi elliptic functions, not only when they are circular functions. The elliptic functions are particularly suitable for cubic oscillators. The application of Chen's procedure is straightforward when the parameter of the Jacobi elliptic function is known *a priori*, as for example when $m=1/2$ for perturbed *pure* cubic oscillators or when $m=0$, i.e., when the Jacobi elliptic function reduces to a circular function ($\mathrm{cn}\,(\psi, m=0)=\cos\,(\psi)$) for perturbed linear oscillators. But for perturbed cubic oscillators the elliptic parameter is unknown, and then one must add another additional Galerkin condition (corresponding to this parameter) to that given by Chen. The results for the examples studied were good, some of them coinciding with results obtained from the harmonic balance method.

ACKNOWLEDGMENT

The author acknowledges the Comision de Investigacion Cientifica y Tecnica (CICYT), (project PB87-0007), for financial support.

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(*Received* 20 *September* 1988)

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