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A WEIGHTED MEAN-SQUARE METHOD OF "CUBICATION" FOR NON-LINEAR OSCILLATORS

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An alternative to the linearization technique in oscillator problems of the type $\ddot{x}+f(x)=0$ is studied in which these equations are approximated by others of cubic type $\ddot{x}+a_3x^3=0$ using a weighted mean-square method. This "cubication" method is checked by studying the period, energy and solution for several examples. For all forces with odd polynomials and a positive largest coefficient, cubication is more accurate than linearization for large amplitudes, and for all amplitudes if $f(x)$ has a cubic and no linear term. For the softening-hardening cubic forces, cubication gives better results than linearization. However, for the softening cubic forces and the flattening springs examined here, linearization is more suitable than cubication.

1. INTRODUCTION

In the linearization technique, an "associate" oscillator

$$\ddot{x}+f^*(x,\dot{x})=0 \tag{1.1}$$

is obtained from the non-linear oscillator problem

$$\ddot{x}+f(x,\dot{x})=0, \tag{1.2}$$

in such a way that the solution of equation (1.1) is an approximate solution of equation (1.2). The term linearization is used because a linear form is chosen for $f^*(x,\dot{x})$. So, for non-dissipative oscillators,

$$\ddot{x}+f(x)=0, \tag{1.3}$$

where the non-linear function $f(x)$ is odd (as will be used throughout the following), linearization applied to equation (1.3) yields the associate oscillator

$$\ddot{x}+f_i(x)=0, \tag{1.4}$$

where

$$f_i(x)=a_ix^i \tag{1.5}$$

and $i=1$ always. This choice of $i=1$ is because the new equation (1.4) has a simple oscillating solution in terms of circular functions

$$x(t)=A\cos(\omega t+\phi). \tag{1.6}$$

Nevertheless, the choice $i=3$ for equation (1.4) with the function (1.5) also has a simple solution

$$x(t)=A\,\mathrm{cn}\,(\omega t+\phi,\tfrac{1}{2}), \tag{1.7}$$

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where cn (u, k^2) is the Jacobi elliptic function cos (am (u, k^2)), A is the amplitude, ω the frequency, ϕ the phase and $\frac{1}{2}$ is the value of the elliptic function parameter k^2. It is then natural to try expression (1.4) with the force (1.5) and $i=3$ as the associate oscillator to the non-linear problem (1.3): i.e., to try a "cubication" technique.

One of us has already worked with a harmonic balance method [1]—using equation (1.7) as trial solution of equation (1.3)—that can be considered a cubication method.

Methods of linearization or cubication differ in the form of obtaining the parameter a_i that defines $f_i(x)$. The criterion is that this a_i must give the best approximation to the period (the period criterion). A complementary criterion might be that the optimal a_i gives a good approximation to the energy of oscillation. This complementary criterion can be justified in the following way. One can rewrite equation (1.3) as

$$\ddot{x}+a_i x^i+R_i(x)=0, \tag{1.8}$$

where

$$R_i(x)=f(x)-f_i(x). \tag{1.9}$$

Equations (1.3) and (1.4) are similar and therefore so are their solutions, when $R_i(x)\simeq 0$. If the forces $f(x)$ and $f_i(x)$ are similar and intersect only at $x=0$ in the oscillation interval, then

$$E=\int_0^A f(x)\,\mathrm{d}x \qquad \text{and} \qquad E_i=\int_0^A f_i(x)\,\mathrm{d}x \tag{1.10}$$

must be similar. Therefore, the solutions of equations (1.3) and (1.4) are similar when $E\simeq E_i$. This is the energy criterion. If $f(x)$ and $f_i(x)$ intersect at several points in the oscillation interval, the energy criterion could be inappropriate because there exist oscillators with very different forces but similar energies. Notice that, in the same way, the period criterion is not appropriate if the oscillators have very different forces but the same or similar period.

2. THE WEIGHTED MEAN-SQUARE CUBICATION METHOD

In the methods of weighted mean-square linearization ($i=1$) or cubication ($i=3$) a search is made over the a_i to minimize

$$\int_{-A}^{A}[f(x)-f_i(x)]^2 w(x)\,\mathrm{d}x. \tag{2.1}$$

One then obtains

$$a_i(A)=(1/N)\int_{-A}^{A} x^i f(x)w(x)\,\mathrm{d}x \tag{2.2}$$

with

$$N=\int_{-A}^{A} x^{2i}w(x)\,\mathrm{d}x. \tag{2.3}$$

Different choices of the weight $w(x)$ give different values of a_i. In the case $i=1$ (linearization) several techniques have been used [2–7]. Denman [2] used $w(x)=(1-x^2/A^2)^{-1/2}$, which is equivalent to making $f_1(x)$ equal to the linear Chebyshev polynomial approximation to $f(x)$. Denman and Howard [3] and Denman and Liu [4] used $w(x)=(1-x^2/A^2)^{\lambda-1/2}$, which is equivalent to taking $f_1(x)$ as the linear ultraspherical polynomial approximation to $f(x)$. Sinha and Srinivasan [5] proposed $w(x)=|x|^{m_1}$ for small oscillations, where the constant m_1 is fixed by the period criterion. Mittal [6] showed that the best choice of m_1 depends on the type of oscillator, and Agrwal and Denman [7] extended the method to large oscillations.

Denman and Liu [4] used a method similar to our cubication technique taking $f_i(x)=a_1x+a_3x^3$. In the present work we use the technique of Sinha and Srinivasan [5], with weight function $w(x)=|x|^{m_3}$ in equations (2.2) and (2.3) with $i=3$. The value of m_3 is chosen to make the period τ_3 of the associate oscillator and the original period τ as close as possible (period criterion). This will be referred to in the following as the cubication method. We show that m_3 (in analogy to m_1 [5, 7]) depends on the form of $f(x)$ and on the amplitude of the oscillations.

The exact expression for the oscillator period is given by

$$\tau=2^{3/2}\int_0^A [V(A)-V(x)]^{-1/2}\,\mathrm{d}x, \tag{2.4}$$

where

$$V(x)=\int_0^x f(\xi)\,\mathrm{d}\xi \tag{2.5}$$

is the potential energy of the oscillator. As expression (1.7) is the solution of equation (1.4) with $i=3$, the period calculated by the cubication method is

$$\tau_3=4\mathrm{K}(\tfrac{1}{2})/(Aa_3^{1/2}), \tag{2.6}$$

where $\mathrm{K}(\frac{1}{2})=1{\cdot}85407\ldots$ is the complete elliptic integral of the first kind with parameter $k^2=\frac{1}{2}$. The total energy of oscillation $E=V(A)$ given by the cubication method is

$$E_3=a_3A^4/4. \tag{2.7}$$

Using the linearization technique gives the period as

$$\tau_1=2\pi/a_1^{1/2} \tag{2.8}$$

and the total energy of oscillation is

$$E_1=a_1A^2/2. \tag{2.9}$$

In the next sections, we apply the cubication method to several examples. A comparison is made with the numerical results obtained for these oscillators following the linearization techniques of references [5] and [7]. Throughout the paper, the values of m_1 used are taken from references [5] and [7]. In the following, "the linearization method" will refer to the method of these references.

3. POLYNOMIAL NON-LINEARITIES

In this section we consider an oscillator submitted to the force

$$f(x)=c_3x^3+cx^{2n+1}. \tag{3.1}$$

One needs to distinguish two regimes: when $c_3A^3\gg|cA^{2n+1}|$, $c_3>0$, the cubic regime, and the non-cubic regime when $|c_3A^3|\ll cA^{2n+1}$ (in particular, the linear regime if $n=0$).

Following Mittal [6] and Agrwal and Denman [7] one can show, using the period criterion (see Appendix 1), that the weighting function $w(x)$ must be

$$w(x)=|x|^{m_3} \tag{3.2}$$

in both regimes, if the weight function is even and independent of the amplitude.

In the cubic regime, the period is (see Appendix 2)

$$\tau \simeq [4\mathrm{K}/(Ac_3^{1/2})][1-(c\alpha^2 A^{2n-2}/2c_3)] \tag{3.3}$$

with

$$\alpha^2=\begin{cases}[2/(n+1)][\mathrm{E}+S_e(n/2)]/\mathrm{K}, & n \text{ even}\\ [2/(n+1)]\{S_0[(n-1)/2]\}/\mathrm{K}, & n \text{ odd}\end{cases}, \tag{3.4}$$

where

$$S_e(j)=(2\mathrm{E}-\mathrm{K})\sum_{i=1}^{j}\Omega_e(i), \qquad S_e(0)=0, \tag{3.5}$$

$$\Omega_e(i)=\begin{cases}1 & \text{if } i=1\\ \frac{3}{5}\cdots\frac{4i-5}{4i-3} & \text{if } i\geqslant 2\end{cases} \tag{3.6}$$

and

$$S_0(j)=K\sum_{i=0}^{j}\Omega_0(i), \qquad \Omega_0(i)=\begin{cases}1 & \text{if } i=0\\ \frac{1}{3}\cdots\frac{4i-3}{4i-1} & \text{if } i\geqslant 1\end{cases}. \tag{3.7, 3.8}$$

Here K and E are the complete elliptic integrals of the first and second kind, respectively, both with parameter $k^2=\frac{1}{2}(\mathrm{E}=1{\cdot}35064\ldots)$. If one equates this expression for τ with the expression for τ_3 obtained from expressions (2.6), (2.2) and (3.1), one finds (as shown in Appendix 1) that $w(x)$ is given by equation (3.2) with the optimal value for m_3 given by

$$m_3=-7+[\alpha^2(2n-2)]/(1-\alpha^2). \tag{3.9}$$

In the non-cubic regime the period is [7]

$$\tau \simeq (2/A^n)\{\pi/[(n+1)c]\}^{1/2}\{\Gamma[1/(2n+2)]/\Gamma[(n+2)/(2n+2)]\}. \tag{3.10}$$

The exact period τ is equal to the cubic period τ_3 if (see Appendix 1) $w(x)$ is given by equation (3.2) and m_3 (optimal m_3) is given by equation (3.9) but now with

$$\alpha^2=(4\mathrm{K}^2/\pi)(n+1)\{\Gamma^2[(n+2)/(2n+2)]/\Gamma^2[1/(2n+2)]\}. \tag{3.11}$$

The optimal values of m_3 for different n, according to expressions (3.9) and (3.4) for the cubic regime and, according to expressions (3.9) and (3.11) for the non-cubic regime, are given in Table 1.

Let us now consider some examples.

3.1. CASE I, HARDENING CUBIC FORCE: $f(x)=c_1x+c_3x^3$, $c_1>0$, $c_3>0$

The linearization method of Sinha and Srinivasan [5], i.e., equation (2.2) with $i=1$ and $w(x)=|x|^{m_1}$, gives

$$a_1(A)=c_1+[(m_1+3)/(m_1+5)]c_3A^2. \tag{3.12}$$

TABLE 1

Optimal m_3 for each n and range of A

n	Cubic regime		Non-cubic regime	
0	0·0864	$A\to\infty$	−0·6231	$A\to 0$
2	0·5365	$A\to 0$	−0·0284	$A\to\infty$
3	1		−0·0490	
4	1·4174		−0·0624	
5	9/5		−0·0718	
∞			−0·1248	

In the present method (cubication), one uses equation (2.2) with $i=3$ and $w(x)$ given by (3.2) to obtain

$$a_3(A)=[(m_3+7)/(m_3+5)](c_1/A^2)+c_3. \tag{3.13}$$

With the preceding expressions and with equations (2.6)-(2.9) we have evaluated the exact period and energy of the associate oscillators for different amplitudes: $\tau=4K(k^2)/\omega$, $E=c_1A^2/2+c_3A^4/4$, with $k^2=c_3A^2/[2(c_1+c_3A^2)]$, and $\omega^2=c_1+c_3A^2$.

For this oscillator, $m_1=3$ is the optimal value for small oscillations [5] and $m_1=2{\cdot}0864$ is the optimal value for large oscillations [7]. The values for m_3 were taken from Table 1: $m_3=0{\cdot}0864$ for small oscillations and $m_3=-0{\cdot}6231$ for large oscillations. The simpler values $m_1=2$ and $m_3=0$ are, nevertheless, very good both here and in many other cases, as will be seen in following examples.

The relative errors of the period and the energy versus the non-linearity factor $\nu=c_3A^2/c_1$ are shown in Figure 1. These figures show that the results are better with cubication than with linearization, except for small non-linearity factors.

3.2. CASE II, SOFTENING CUBIC FORCE: $f(x)=c_1x+c_3x^3$, $c_1>0$, $c_3<0$

For oscillatory motions, the non-linearity factor ν is between −1 and 0, and therefore there are no large oscillations in this case. Expressions (3.12) and (3.13) are valid here as well. The values of τ and E are as in the case of the hardening cubic force, but now

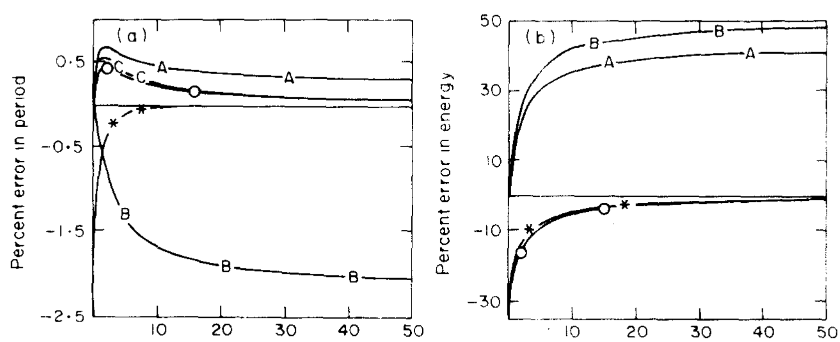

Figure 1. Relative error in period (a), and energy (b), *vs.* non-linearity factor for the oscillator $\ddot{x}+c_1x+c_3x^3=0$, $c_1>0$, $c_3>0$. The parameters are $m_1=2$, A; $m_1=3$, B; $m_1=2{\cdot}0864$, C $\simeq$ A in (b); $m_3=0$, O; $m_3=0{\cdot}0864$, + $\simeq$ C in (a) and + $\simeq$ O in (b); $m_3=-0{\cdot}6231$, *. Here and in the following figures we use the following conventions: B for the optimal m_1 when $A\to 0$; C for the optimal m_1 when $A\to\infty$; + for the optimal m_3 when $A\to 0$; and * for the optimal m_3 when $A\to\infty$; symbol$_1$ $\simeq$ symbol$_2$ means that the line corresponding to symbol$_1$ is given by the line corresponding to symbol$_2$ because they are the same or very close.

with $k^2 = -c_3A^2/(2c_1 + c_3A^2)$ and $\omega^2 = c_1 + (c_3A^2)/2$. The optimal values for $A \to 0$ are also the same as for case I.

The relative errors of τ_1, τ_3, E_1 and E_3 versus ν are shown in Figure 2: one sees that the linearization method is better over the total range of amplitudes.

3.3. CASE III, SOFTENING-HARDENING CUBIC FORCE: $f(x) = c_1x + c_3x^3$, $c_1 < 0$, $c_3 > 0$

In this case, for symmetrical oscillations, $\nu < -2$. The exact values of τ and E are given by the same expressions as in case I. The relative error of the period and energy for the associate oscillator versus the non-linearity factor ν are shown in Figure 3. For $\nu \geqslant -3$ neither method is good, but the cubication method is better for $\nu < -3$.

3.4. CASE IV, ODD-POWER FORCE: $f(x) = x + x^3 + x^5 + x^7$

In general, if $f(x) = \sum_j x^{2j+1}$ equations (2.2) and (2.3) give

$$a_1(A) = \sum_j [(m_1+3)/(m_1+2j+3)]A^{2j}, \tag{3.14}$$

$$a_3(A) = \sum_j [(m_3+7)/(m_3+2j+5)]A^{2j-2}. \tag{3.15}$$

The relative errors of the period and energy of the associate oscillators versus the amplitude for different values of m_i are shown in Figure 4. The value of τ was calculated by numerical

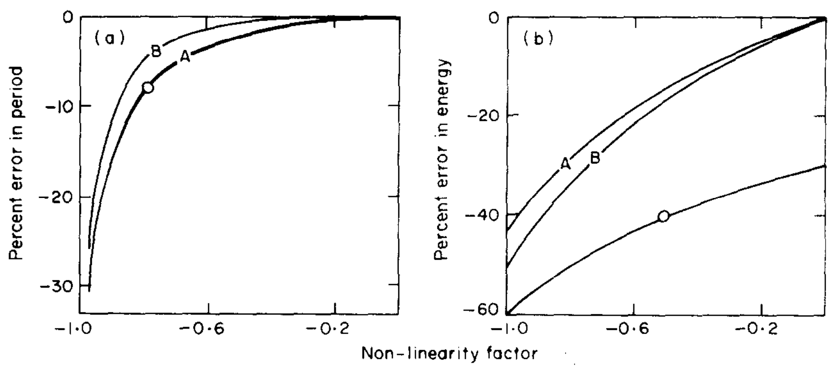

Figure 2. Relative error in period (a), and energy (b), *vs.* non-linearity factor for the oscillator $\ddot{x} + c_1x + c_3x^3 = 0$, $c_1 > 0$, $c_3 < 0$. The parameters are $m_1 = 2$, A; $m_1 = 3$, B; $m_3 = 0$, ○; $m_3 = 0{\cdot}0864$, + ≃ ○ in (a) and in (b).

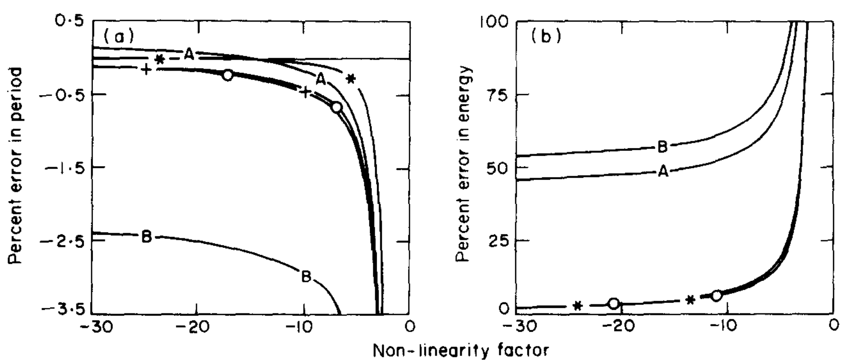

Figure 3. Relative error in period (a), and energy (b), *vs.* non-linearity factor for the oscillator $\ddot{x} + c_1x + c_3x^3 = 0$, $c_1 < 0$, $c_3 > 0$. The parameters are $m_1 = 2$, A; $m_1 = 3$, B; $m_1 = 2{\cdot}0864$, C ≃ + in (a) and C ≃ A in (b); $m_3 = 0$, ○; $m_3 = 0{\cdot}0864$, + ≃ ○ in (b); $m_3 = -0{\cdot}6231$, *.

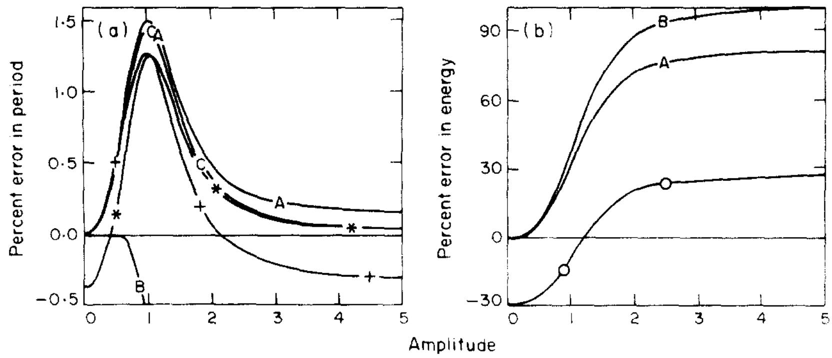

Figure 4. Relative error in period (a), and energy (b), *vs.* amplitude for the oscillator $\ddot{x}+x+x^3+x^5+x^7=0$. The parameters are $m_1=2$, A; $m_1=3$, B; $m_1=2{\cdot}0212$, C $\simeq$ A in (b); $m_3=0$, O; $m_3=0{\cdot}0864$, + $\simeq$ O in (b); $m_3=-0{\cdot}0490$, * $\simeq$ O in (b). The line corresponding to $m_3=0$ is not plotted in (a). It lies between lines * and +.

integration. In the linearization method, $m_1=3$ is the best value in the linear regime ($A\simeq 0$), and $m_1=2{\cdot}0212$ is the best for large A. In the cubication method, the best m_3 is 0·0864 for the linear regime ($A\to 0$, $n=0$, Table 1) and $m_3=-0{\cdot}0490$ for the non-cubic regime ($A\to\infty$, $n=3$).

This case is well suited to showing how the energy criterion explained in section 1 works. The numerical solutions of this problem with $f(x)=x+x^3+x^5+x^7$ for different values of the amplitude versus the normalized time (t/τ) are shown in Figure 5. Solutions of the associate linear and cubic oscillators are also given in this figure. Errors in the period of the linear and cubic solution are small (see Figure 4(a)) and have been taken to be zero in order to simplify the plot. For $A=0{\cdot}5$ the difference between the exact and the linear energy of the oscillator (E and E_1) is small (see Figure 4(b)) and less than that between the exact and the cubic oscillator (E_3). It is shown in Figure 5 that the linear solution is closer to the exact one for this case. For $A=1{\cdot}2$, the error in the energy is small when calculated with the cubication method, and less than that given by the linearization technique (see Figure 4). When $A\geqslant 3$, although the error in the energy is

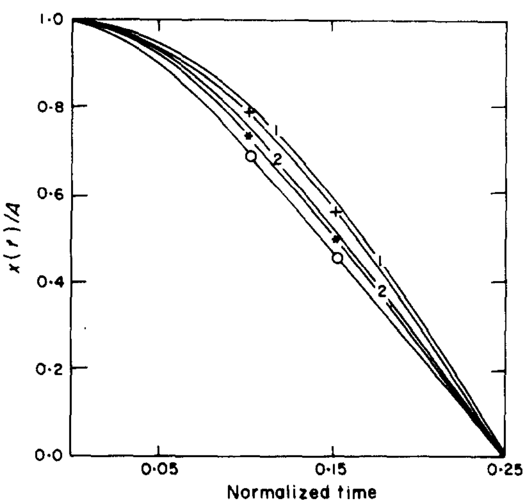

Figure 5. Numerical solution for the oscillator $\ddot{x}+x+x^3+x^5+x^7=0$ with initial conditions $\dot{x}(0)=0$ and $x(0)=A$, where $A=0{\cdot}5$, +; $A=1{\cdot}2$, *; and $A\geqslant 3$, O. The approximate linear solution $A\cos(\omega t)$, labeled 1, and the approximate cubic solution $A\,\mathrm{cn}(\omega t, 1/2)$, labeled 2, are also given.

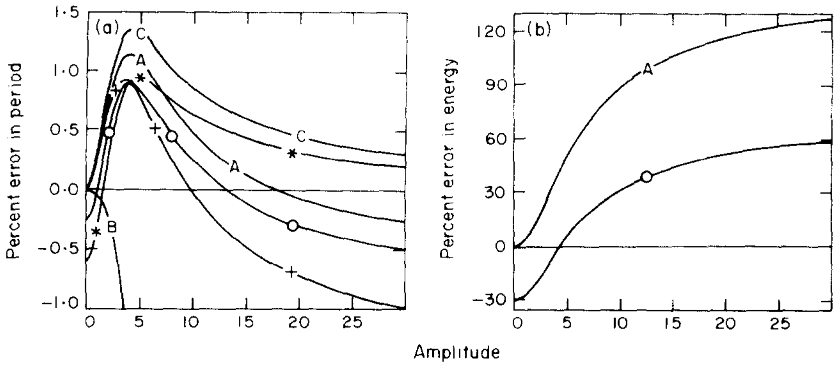

Figure 6. Relative error in period (a), and energy (b), *vs.* amplitude for the oscillator $\ddot{x}+\sinh x=0$. The parameters are $m_1=2$, A; $m_1=3$, B; $m_1=1\cdot9348$, C $\simeq$ A in (b); $m_3=0$, O; $m_3=0\cdot0864$, + $\simeq$ O in (b); $m_3=-0\cdot1248$, * $\simeq$ O in (b).

large with either method, it is lower with cubication. For large amplitudes ($A\geqslant 3$) the difference between the two solutions is practically constant, which is consistent with the practically constant percentage error in energy for large A (see Figure 4(b)).

4. SINH NON-LINEARITY

We now consider $f(x)=\omega_0^2\sinh x$. This function can be represented by the power series

$$\sinh x = x+(x^3/6)+\sum_{n=2}^{\infty}[x^{2n+1}/(2n+1)!]. \tag{4.1}$$

Therefore the optimal m_3's for small amplitudes ($A\to 0$ with $n=0$) and for large amplitudes ($A\to\infty$ with $n\to\infty$) are those given in Table 1. Hence, for small oscillations one has $m_1=3$ [5] and $m_3=0\cdot0864$; for large oscillations $m_1=1\cdot9348$ [7] and $m_3=-0\cdot1248$. It is possible to obtain the best value of m_3 when $A\to\infty$ in the following different way (as shown in reference [7] for m_1). The exact period of oscillation is [4]

$$\tau=(4/\omega_0)\,\mathrm{sech}\,(A/2)\mathrm{K}(k^2), \tag{4.2}$$

with $k=\tanh(A/2)$. For large A

$$\tau\simeq(4/\omega_0)A\exp(-A/2). \tag{4.3}$$

Using the cubication method, one has

$$\tau_3\simeq(4\mathrm{K}/\omega_0)[2/(m_3+7)]^{1/2}A\exp(-A/2) \tag{4.4}$$

for large values of A. The best m_3 is easily calculated by equating expressions (4.3) and (4.4). Then $m_3=2\mathrm{K}^2-7=-0\cdot1248\ldots$.

Plots of the relative errors for the period and energy ($E=V(A)=\omega_0^2[\cosh(A)-1]$) for the above values of m_i and for $m_1=2$ and $m_3=0$ are shown in Figure 6 (as the relative error for the energy is so large when using $m_1=3$, this is not shown in Figure 6(b)). One therefore concludes that the method of cubication is better in this case also, except for low values of A.

5. FLATTENING SPRINGS

In this type of oscillator the force tends to a constant value when $x\to\infty$. An example is $f(x)=\omega_0^2\tanh x$.

For small amplitudes the best value of m_3 depends on the particularities of each oscillator, but for large amplitudes the best m_3 is the same for all flattening springs. following Agrwal and Denman [7] if $A\to\infty$, when $x\to\infty$, $f(x)\to\pm\omega_0^2$, $V(x)\to\omega_0^2|x|+C$ and, then, from equation (2.4),

$$\tau\simeq(8^{1/2}/\omega_0)\int_0^A(A-x)^{-1/2}\,dx=(4/\omega_0)(2A)^{1/2}. \tag{5.1}$$

By means of the cubication method, for $A\to\infty$

$$\tau_3\simeq(4K/\omega_0)A^{1/2}[(m_3+4)/(m_3+7)]^{1/2}. \tag{5.2}$$

Equating (5.1) and (5.2), one obtains the value of m_3 that gives the correct behaviour of the period:

$$m_3=(14-4K^2)/(K^2-2)=0{\cdot}1736\ldots. \tag{5.3}$$

As $\tanh x=x-x^3/3+\cdots$ the best values for $A\to 0$ are $m_3=0{\cdot}0864$ (Table 1) and $m_1=3$. For $A\to\infty$ the best value is [7] $m_1=2{\cdot}27898\ldots$.

The relative errors for the period and energy ($E=V(A)=\omega_0^2\ln[\cosh(A)]$) are shown in Figure 7. As the relative error for the energy is large when using $m_1=3$, this case is not plotted in Figure 7(b). Linearization is better in this case, for any amplitude.

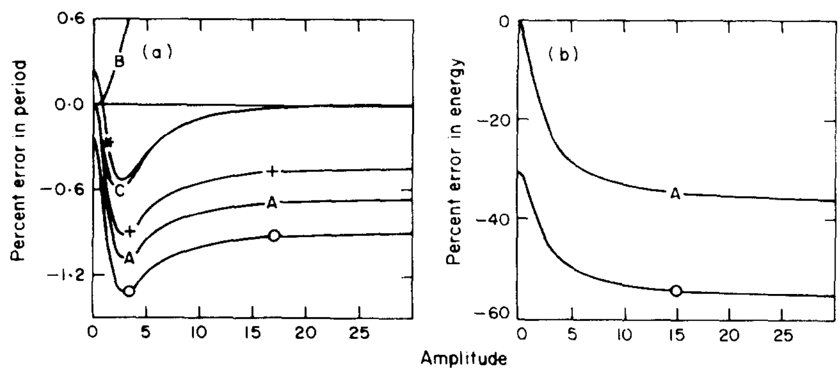

Figure 7. Relative error in period (a), and energy (b), *vs.* amplitude for the oscillator $\ddot{x}+\omega_0^2\tanh x=0$. The parameters are $m_1=2$, A; $m_1=3$, B; $m_1=2{\cdot}27898$, C $\simeq$ A in (b); $m_3=0$, ○; $m_3=0{\cdot}0864$, + $\simeq$ ○ in (b); $m_3=0{\cdot}1736$, * $\simeq$ ○ in (b).

6. DISCUSSION

Sinha and Srinivasan [5], Mittal [6] and Agrwal and Denman [7] have developed a very good weighted mean-square method of linearization ($w(x)=|x|^{m_1}$) for odd-power forces in non-dissipative oscillators. They obtained the best values of m_1 to approximate the period in two regimes: for large oscillations, and for small oscillations if the force has a non-zero linear term.

The present paper has described a similar method, but now of cubication, in which the best m_3 in the weight function $w(x)=|x|^{m_3}$ is obtained for large oscillations, and for small oscillations if $f(x)$ has a non-zero cubic term.

The results (associated solution, period, and energy) are good, and a clear improvement over linearization for certain types of oscillators. If the force $f(x)$ is closer in form to a_3x^3 than to a_1x, the energy is better approximated by the associate cubic oscillator than by the associate linear oscillator. For all odd polynomials $f(x)$ with a positive largest (nth) coefficient, cubication is preferable to linearization for large amplitudes A, and for all A if $f(x)$ has a cubic and no linear term. If n and A tend to infinity, the correct asymptotic period approximation is given when $m_3 = 2K^2 - 7$. This same m_3 gives the best period for large amplitude when $f(x)$ is of the type sinh x. For the softening-hardening cubic forces (negative linear term, positive cubic term) cubication is better than linearization for any amplitude. However, we found that for softening and flattening springs, linearization is more suitable for calculating the best associate solution, period and energy.

Finally, the results for $m_3 = 0$ are similar to those obtained by taking the best m_3, in the non-cubic regime. The choice $m_3 = 0$, when $\int_0^A x^3 f(x)\,dx$ can be written as a sum of elementary functions, gives therefore a simple cubic approximation for the solution and the period.

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REFERENCES

1. S. BRAVO YUSTE and J. DÍAZ BEJARANO 1986 *Journal of Sound and Vibration* **110**, 347-350. Construction of approximate analytical solutions to a new class of non-linear oscillator equations.
2. H. H. DENMAN 1959 *American Journal of Physics* **27**, 524-528. Amplitude dependence of frequency in a linear approximation to the simple pendulum equation.
3. H. H. DENMAN and J. E. HOWARD 1964 *Quarterly of Applied Mathematics* **21**, 325-330. Application of ultraspherical polynomials to non-linear oscillations. I. Free oscillation of the pendulum.
4. H. H. DENMAN and Y. K. LIU 1965 *Quarterly of Applied Mathematics* **22**, 273-292. Application of ultraspherical polynomials to non-linear oscillations. II. Free oscillations.
5. S. C. SINHA and P. SRINIVASAN 1971 *Journal of Sound and Vibration* **16**, 139-148. A weighted mean square method of linearization in non-linear oscillations.
6. A. K. MITTAL 1978 *Journal of Sound and Vibration* **57**, 464-467. Note on weighted mean square linearization method in non-linear oscillation problems.
7. V. P. AGRWAL and H. H. DENMAN 1985 *Journal of Sound and Vibration* **99**, 463-473. Weighted linearization technique for period approximation in large amplitude non-linear oscillations.
8. P. F. BYRD and M. D. FRIEDMAN 1971 *Handbook of Elliptic Integrals for Engineers and Scientists.* Berlin: Springer-Verlag.

APPENDIX 1

We demonstrate here that if $f(x)$ is given by equation (3.1) then the period criterion leads to the weight function $w(x) = |x|^{m_3}$ in both the cubic and the non-cubic regimes.

The approximate period τ_3 of the method of this paper is given by equations (2.6) and (2.2) with $i = 3$. The exact period τ is given by equation (3.3) for the cubic regime and by equation (3.10) for the non-cubic regime. Equating τ^2 and τ_3^2, and taking $w(x)$ as even, one finds

$$\int_0^A x^{2n+4} w(x)\,dx = \alpha^2 A^{2n-2} \int_0^A x^6 w(x)\,dx, \tag{A1}$$

where α^2 is given by equation (3.4) for the cubic regime and by equation (3.11) for the non-cubic regime. This equation is valid for all values of A (within each regime). Then, differentiating with respect to A, one obtains

$$(1-\alpha^2)[\alpha^2(2n-2)]^{-1}A^7 w(A)=\int_0^A x^6 w(x)\,\mathrm{d}x. \tag{A2}$$

Differentiating again and replacing A by x, one finds

$$\mathrm{d}w/w=m_3(\mathrm{d}x/x), \tag{A3}$$

where m_3 is expressed by equation (3.9). The weight function is then given by equation (3.2), with m_3 depending only on n in both regimes.

APPENDIX 2

We show here the derivation of equation (3.3) for the period τ in the cubic regime. From equations (3.1), (2.4) and (2.5) one finds

$$\tau=(32/c_3)^{1/2}\int_0^A\left[1+\frac{4}{2n+2}\frac{c}{c_3}\frac{A^{2n+2}-x^{2n+2}}{A^4-x^4}\right]^{-1/2}(A^4-x^4)^{-1/2}\,\mathrm{d}x. \tag{A4}$$

With the change of variable $y=x/A$, and as $c_3A^3\gg|cA^{2n+1}|$ in the cubic regime, one has

$$\tau\simeq[32/(A^2c_3)]^{1/2}\int_0^1\left[1-\frac{1}{n+1}\frac{c}{c_3}A^{2n-2}\frac{1-y^{2n+2}}{1-y^4}\right](1-y^4)^{-1/2}\,\mathrm{d}y. \tag{A5}$$

For even n

$$(1-y^{2n+2})/(1-y^4)=\sum_{i=1}^{p}y^{4i-2}+1/(1+y^2), \tag{A6}$$

with $p=n/2$. For odd n

$$(1-y^{2n+2})/(1-y^4)=\sum_{i=0}^{q}y^{4i}, \tag{A7}$$

with $q=(n-1)/2$. Following reference [8], one has

$$\int_0^1(1-y^4)^{-1/2}\,\mathrm{d}y=2^{-1/2}\mathrm{K},\qquad \int_0^1(1+y^2)^{-1}(1-y^4)^{-1/2}\,\mathrm{d}y=2^{-1/2}\mathrm{E},$$

$$\int_0^1 y^i(1-y^4)^{-1/2}\,\mathrm{d}y=2^{-1/2}I_i. \tag{A8}$$

Therefore, substituting equations (A6) and (A7) into equation (A5), and defining

$$S_e(j)\equiv\sum_{i=1}^{j}I_{4i-2},\qquad S_0(j)\equiv\sum_{i=0}^{j}I_{4i}, \tag{A9}$$

one has equation (3.3).

It is shown in reference [8] that

$$I_{4i-2}=\frac{3}{5}\cdots\frac{4(i-1)-5}{4(i-1)-3}\frac{4i-5}{4i-3}I_2,\qquad I_{4i}=\frac{1}{3}\cdots\frac{4(i-1)-3}{4(i-1)-1}\frac{4i-3}{4i-1}I_0, \tag{A10}$$

with $I_2=2\mathrm{E}-\mathrm{K}$ and $I_0=\mathrm{K}$. It is then straightforward to show that $S_e(j)$ is given by expression (3.5) and $S_0(j)$ by expression (3.7).