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Extension and improvement to the Krylov–Bogoliubov methods using elliptic functions

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It is shown that the Krylov–Bogoliubov methods that give the approximate oscillatory solution of the equation $\ddot{x} + c_1 x + c_3 x^3 + \varepsilon f(x, \dot{x}, t) = 0$ in terms of Jacobi elliptic functions are applicable not only when $c_1 > 0$ and $c_3 > 0$, but also when $c_1 > 0$ and $c_3 < 0$ or when $c_1 < 0$ and $c_3 > 0$. In particular, the most precise of these methods, the Christopher–Brocklehurst method, is discussed in detail. Its accuracy has been improved by utilizing the transformation properties of elliptic functions with respect to their parameters.

1. Introduction

The Krylov–Bogoliubov (KB) method is a well-known method of obtaining approximate solutions to the class of quasi-linear differential equations

$$\ddot{x} + c_1 x + \varepsilon f(x, \dot{x}, t) = 0 \tag{1}$$

where ε is a small constant coefficient.

Because the solution of (1) with $\varepsilon = 0$—the generating equation—is given in terms of circular functions, it is assumed in this method that (1) has the solution

$$x(t) = A(t) \cos(\omega t - \phi(t)) \tag{2}$$

and where $A(t)$ and $\phi(t)$ are determined with the obvious condition that expression (2) is the solution of (1). This method was extended and justified mathematically by Bogoliubov and Mitropolsky (KBM).

In a series of papers, Barkham and Soudack (1969, 1970) and Soudack and Barkham (1970, 1971) extended these methods to the class of quasi-cubic differential equations

$$\ddot{x} + c_1 x + c_3 x^3 + \varepsilon f(x, \dot{x}, t) = 0 \tag{3}$$

giving the approximate solution in terms of Jacobi elliptic functions

$$x = A(t) \operatorname{cn}(\Psi(t), \mu^2) \tag{4}$$

with $\Psi(t) = \omega t - \phi(t)$—the Barkham–Soudack (BS) method. The functions $A(t)$ and $\phi(t)$ are to be determined in such a way that (4) satisfies (3).

Christopher and Brocklehurst (1974) developed a method (the CB method), also of the Krylov–Bogoliubov type, where expression (4) is taken to be the solution of (3), but now using the amplitude A, parameter μ^2, frequency ω, and phase ϕ time-dependent. That is

$$x(t) = A(t) \operatorname{cn}(\Psi(t), \mu^2(t)) \tag{5}$$

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with

$$\Psi(t) = \int_0^t \omega(t)\, dt - \phi(t)$$

The authors of these previously mentioned methods consider them applicable to (3) only when the oscillator is hard, i.e. when $c_1 > 0$ and $c_3 > 0$ (oscillator 1). However, there is nothing in the methods that prevents their application to hard–soft cubic oscillators—$c_1 > 0$, $c_3 < 0$ (oscillator 2)—or to the soft–hard cubic ones—$c_1 < 0$, $c_3 > 0$ (oscillator 3).

One can understand that the methods are not believed valid for soft–hard oscillators when $A^2 < -c_1/c_3$ or for hard–soft ones, since μ^2 is then either larger than one or negative, and it is not usual to work with elliptic functions having parameters in this range. However, more suprising is that the aforementioned authors do not apply their methods to the soft–hard oscillator when $A^2 > -c_1/c_3$, since then $1/2 < \mu^2 < 1$. In fact, for this case, every relation obtained by the aforementioned authors using the methods mentioned above is obviously valid.

In this paper the following are carried out.

(i) We show how to apply the BS and CB methods to each oscillator type, and how the most accurate of them, the CB method, can be made even more accurate by utilizing the transformation properties of the elliptic functions.

(ii) As an example to check the methods, (3) is used with $\varepsilon f(x, \dot{x}) = b\dot{x}$, i.e.

$$\ddot{x} + c_1 x + c_3 x^3 + b\dot{x} = 0 \tag{6}$$

because this is an important example used in all the papers cited (Barkham and Soudack 1969, 1970, Christopher and Brocklehurst 1974, Soudack and Barkham 1970, 1971), and its simplicity allows one to see clearly how the methods work and the validity of the approximations used.

(iii) The determination of $A(t)$ is fundamental in these methods, since the amplitude gives the oscillator's energy, frequency, period (i.e. μ^2) and, as in the references, $x(t)$. The accuracy of each technique of solution is tested by simply comparing the oscillation amplitude that it gives with that numerically evaluated.

Before discussing the BS and CB methods, we review some properties of elliptic functions.

2. Properties of elliptic functions

As is well known, the solution to (3) with $\varepsilon = 0$ (generating equation) is given by

$$x(t) = A \operatorname{cn}(\omega t - \phi, \mu^2) \tag{7}$$

with

$$\omega^2 = c_1 + c_3 A^2 \tag{8}$$

$$\mu^2 = \frac{c_3 A^2}{2(c_1 + c_3 A^2)} \tag{9}$$

where A and ϕ are constants determined by the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$. In particular, when $\dot{x}_0 = 0$ then $A = x_0$ (and $\phi = 0$).

The oscillator's energy is defined by $E = \dot{x}^2 + V(x)$ with $V(x) = c_1 x^2 + c_3 x^4/2$.

Sometimes it is useful to have (8) and (9) in terms of the non-linearity factor $v = c_3 A^2/c_1$, i.e.

$$\omega^2 = c_1(1+v)$$

$$\mu^2 = \frac{v}{2(1+v)}$$

For oscillator 1, $v > 0$, i.e. $0 < \mu^2 < 1/2$. For oscillator 2, $-1 < v < 0$, i.e. $\mu^2 < 0$. Finally, for oscillator 3, $v < -1$: if $-2 < v < -1$ (i.e. $E < 0$) then $\mu^2 > 1$, and if $v < -2$ ($E > 0$) then $1/2 < \mu^2 < 1$. One problem is that tables do not give the values of elliptic functions when the parameter μ^2 is outside the interval [0, 1]. However, this is easy to solve by means of parameter transformations. If $\mu^2 < 0$, by the negative parameter transformation (Abramowitz and Stegun 1972)

$$\text{cn}(\Psi, \mu^2) = \text{cd}(\Psi_N, \sigma^2) \tag{10}$$

with

$$\sigma^2 = -\frac{\mu^2}{1-\mu^2} \tag{11}$$

$$\Psi_N = \frac{\Psi}{\sigma_1} \tag{12}$$

and $\sigma_1 = (1-\sigma^2)^{1/2}$. From (11) it is obvious that σ^2 lies between 0 and 1 if $\mu^2 < 0$, as we require. Also, for the first and second complete elliptic integrals:

$$\left.\begin{aligned} K(\mu^2) &= \sigma_1 K(\sigma^2) \\ E(\mu^2) &= E(\sigma^2)/\sigma_1 \end{aligned}\right\} \tag{13}$$

If $\mu^2 > 1$, from the reciprocal parameter transformation or the jacobian real transformation (see Abramowitz and Stegun 1972)

$$\text{cn}(\Psi, \mu^2) = \text{dn}(\Psi_R, \eta^2) \tag{14}$$

$$\text{sn}(\Psi, \mu^2) = \eta\,\text{sn}(\Psi_R, \eta^2) \tag{15}$$

with

$$\eta = \frac{1}{\mu} \tag{16}$$

and

$$\Psi_R = \frac{\Psi}{\eta} \tag{17}$$

Of course from (16) η^2 is between 0 and 1 when $\mu^2 > 1$. Also (see Erdélyi 1981)

$$K(\mu^2) = \eta(K + iK') \tag{18}$$

$$E(\mu^2) = \frac{E - \eta_1^2 K + i(E' - \eta^2 K')}{\eta} \tag{19}$$

where $\eta_1^2 = 1 - \eta^2$, $E = E(\eta^2)$, $K = K(\eta^2)$, and where $K' = K'(\eta^2)$ and $E' = E'(\eta^2)$ are the first and second complementary complete elliptic integrals, respectively.

Finally, by means of the Gauss transformation or the descending Landen transformation (see Abramowitz and Stegun 1972) we can decrease the μ^2 parameter when $0 < \mu^2 < 1$. Then

$$\mathrm{cn}\,(\Psi, \mu^2) = \frac{cndn}{(1 + \xi sn^2)}$$

with $cn = \mathrm{cn}\,(\Psi_G, \xi^2)$, $dn = \mathrm{dn}\,(\Psi_G, \xi^2)$ and $sn = \mathrm{sn}\,(\Psi_G, \xi^2)$, and where

$$\xi = \frac{1 - \mu_1}{1 + \mu_1} \tag{20}$$

$$\Psi_G = \frac{\Psi}{1 + \xi} \tag{21}$$

Of course $\mu_1^2 = 1 - \mu^2$. Also

$$\left.\begin{aligned} K(\mu^2) &= (1 + \xi)K(\xi^2) \\ E(\mu^2) &= \left[\frac{2E(\xi^2)}{1 + \xi}\right] - (1 - \xi)K(\xi^2) \end{aligned}\right\} \tag{22}$$

3. Barkham and Soudack method

In this method, which is analogous to the KBM method, we take

$$x(t) = a(t)\ \mathrm{cn}\,(\Psi(t), \mu_0^2) \equiv acn \tag{23}$$

as the solution of (3) with $\Psi(t) = \omega t + \phi(t)$, $\omega^2 = c_1 + c_3 a^2$, $\mu_0^2 = c_3 a_0^2/[2(c_1 + c_3 a_0^2)]$, $a_0 = a(0)$, and where the amplitude $a(t)$ and the phase $\phi(t)$ satisfy

$$\dot{a} = \beta A^{(1)} + \beta^2 A^{(2)} + \ldots$$

$$\dot{\phi} = \beta B^{(1)} + \beta^2 B^{(2)} + \ldots$$

with β small. The functions $A^{(i)}$, $B^{(i)}$ are to be chosen in such a way that (23) is a solution of (3). In the first approximation $A^{(1)}$ and $B^{(1)}$ are determined by

$$f(x, \dot{x}, t) = sndn[a\dot{B}^{(1)} + 2A^{(1)}(c_1 + c_3 a^2)^{1/2} + T_1] - cn\left[\frac{\dot{A}^{(1)} - (c_1 + \delta c_3 a^2)(2aB^{(1)}T_2 + T_3)}{c_1 + c_3 a^2}\right]$$

If we assume that ω is constant (the unrefined approximation; see Barkham and Soudack 1970) then $T_1 = 0$, $T_2 = 1$, and $T_3 = 0$. If ω is time dependent (the refined approximation; see Soudack and Barkham 1970) then

$$T_1 = \frac{2c_3 a^2 A^{(1)} + c_3 a^2 t\dot{A}^{(1)}}{(c_1 + c_3 a^2)^{1/2}}$$

$$T_2 = (c_1 + c_3 a^2)^{1/2}$$

$$T_3 = 2c_3 a^2 t A^{(1)}$$

The constant δ is defined as

$$\delta = \frac{\int_{-K}^{K} \mathrm{cn}^3(\Psi)\, d\psi}{\int_{-K}^{K} \mathrm{cn}\,(\Psi)\, d\Psi} \tag{24}$$

with $K = K(\mu^2)$ if $0 < \mu^2 < 1$. If $\mu^2 > 1$, δ can be calculated by replacing cn with dn in (24) with $K = K(\eta^2)$—see (14). If $\mu^2 < 0$, cn must be replaced with cd in (24) with $K = K(\sigma^2)$.

Applying this method to (6), which we use as an example, we obtain (as in Barkham and Soudack 1970, Soudack and Barkham 1970) that

$$A_u(t) \equiv a(t) = a(0) \exp\left(\frac{-bt}{2}\right) \tag{25}$$

for the unrefined approximation, and

$$A_r(t) \equiv a(t) = a(0) \exp\left(\frac{-bt}{2(1+\mu_0^2)}\right) \tag{26}$$

for the refined approximation.

A more detailed analysis of this method would convince us that there is nothing that would inherently prevent its application to oscillators 2 and 3. That is, in our example, the amplitude is given according to the BS method using (25) and (26) not only for $c_1 > 0$ and $c_3 > 0$, but also for $c_1 > 0$ and $c_3 < 0$ and also $c_1 < 0$ and $c_3 > 0$. Although we do not make this analysis here, it is made for the more precise method that we study in the next section.

4. Christopher–Brocklehurst method

The first version, as expounded by Christopher (1973), was valid only for solving (6) with the restriction $c_1 > 0$ and $c_3 > 0$, i.e. for oscillator 1. Following this technique, Bravo Yuste and Díaz Bejarano (1987) applied the method to oscillators 2 and 3. Christopher and Brocklehurst (1974) extended the version given by Christopher (1973) to solving the more general (3). This method is more accurate than the BS method basically because it allows the elliptic function parameter of the solution to change with time.

solution of (3). This solution is chosen since (7) is the solution (generating solution) of (3) with $\varepsilon = 0$ (generating equation).

Apart from expression (5) being the solution of (3) (constraint 1), it is convenient to impose on $A(t)$, $\omega(t)$, $\phi(t)$, and $\mu^2(t)$ three additional constraints, as follows:

$$\text{constraint 2:} \quad \dot{x}(t) = A\omega cn_{\Psi} \tag{27}$$

and—see (8) and (9)

$$\text{constraint 3:} \quad \omega^2 = c_1 + c_3 A^2 \tag{28}$$

$$\text{constraint 4:} \quad \mu^2 = \frac{c_3 A^2}{2(c_1 + c_3 A^2)} \tag{29}$$

The notation is of course $f_{\beta}(\alpha, \beta) = \partial f / \partial \beta$. The task of obtaining the solution $x(t)$ is thus transformed into one of obtaining the functions $A(t)$, $\omega(t)$, $\phi(t)$, and $\mu^2(t)$ that satisfy the four constraints.

Differentiating (5) with respect to t and from (27):

$$\dot{A}cn - A\dot{\phi}cn_{\psi} + A\dot{\mu}cn_{\mu} = 0 \tag{30}$$

Differentiating (27) again, substituting this into (3) and using (28) and (29), we obtain

$$\dot{A}\omega cn_{\psi} + A\dot{\omega}cn_{\psi} - A\omega\dot{\phi}cn_{\psi\psi} + A\omega\dot{\mu}cn_{\psi_{\mu}} + \varepsilon f(Acn, A\omega cn_{\psi}, t) = 0 \tag{31}$$

Taking $\dot{\phi}$ from (30) and substituting it into (31), we find

$$\dot{A}\omega[(cn_{\psi})^2 - cncn_{\psi\psi}] + \dot{\omega}A(cn_{\psi})^2 + \dot{\mu}A\omega[cn_{\psi_{\mu}} - cn_{\mu}cn_{\psi\psi}] + \varepsilon fcn_{\psi} = 0 \tag{32}$$

But

$$cn_{\psi} = -sndn$$

$$cn_{\psi\psi} = cn(1 - 2dn^2)$$

$$cn_{\psi_{\mu}} - cn_{\mu}cn_{\psi\psi} = \left[\frac{\mu}{1-\mu^2}\right][sn^2dn^2(1 - 2sn^2) + sn^2cn^2(1 - 2dn^2)]$$

Also, from (28) and (29),

$$\frac{\dot{\omega}}{\omega} = \frac{2\dot{\mu}\mu}{1 - 2\mu^2} \tag{33}$$

Putting these relations into (32), we obtain

$$(1 - 2\mu^2 sn^2 + \mu^2 sn^4)\left(\frac{\dot{A}}{A}\right) + \left(sn^2 - \frac{sn^4}{2}\right)\left(\frac{\dot{\omega}}{\omega}\right) = \left(\frac{\varepsilon}{A\omega}\right)fsndn \tag{34}$$

In short, we must find four functions $A(t)$, $\omega(t)$, $\phi(t)$, and $\mu^2(t)$ that satisfy (28)–(31) or, equivalently, (28)–(30) and (34). This is not easy. However, the coefficients of $\dot{A}/A$, $\dot{\omega}/\omega$ and f in (34) are roughly periodic and therefore we can reduce this expression to a simpler form by using the averaging principle method. So, (34) becomes

$$\frac{\dot{A}}{A} + R_1(\mu^2)\frac{\dot{\omega}}{\omega} = -\left(\frac{\varepsilon}{2A\omega}\right)\frac{\langle fsndn\rangle}{(Q_1 - \mu^2 Q_2)} \tag{35}$$

with

$$R_1(\mu^2) = \frac{Q_1 - \frac{1}{2}Q_2}{2(Q_1 - \mu^2 Q_2)} \tag{36}$$

$$\left.\begin{aligned} Q_1 &= \langle sn^2\rangle = \frac{1 - \left(\frac{E}{K}\right)}{\mu^2} \\ Q_2 &= \langle sn^4\rangle = \frac{2 + \mu^2 - 2(1+\mu^2)\left(\frac{E}{K}\right)}{3\mu^4} \end{aligned}\right\} \tag{37}$$

and where $K = K(\mu^2)$, $E = E(\mu^2)$ and

$$\langle \mathrm{pq}(\Psi, \mu^2)\rangle \equiv \frac{1}{4K}\int_0^{4K} \mathrm{pq}(\Psi, \mu^2)\, d\Psi \tag{38}$$

We have used the relationship

$$1 - 2\mu^2 Q_1 + \mu^2 Q_2 = 2(Q_1 - \mu^2 Q_2)$$

Note that the averaging is over the real period of the Jacobi elliptic function T_r which figures in (38) as $4K$: $T_r \equiv 4K$ (the Jacobi elliptic functions are doubly periodic, i.e. periodic along both the real and the imaginary axes).

For our example (6), (35) is given as

$$\frac{\dot{A}}{A} + R_1(\mu^2)\frac{\dot{\omega}}{\omega} = -\frac{b}{2} \tag{39}$$

At this point, following Bravo Yuste and Díaz Bejarano (1987) and Christopher and Brocklehurst (1974) we show two ways to evaluate $A(t)$ that we shall call the simple and sophisticated modes.

Simple mode

Assume that $R_1(\mu^2) \equiv R$ is constant over the integration interval. Then the integration of (39) is simple:

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\frac{\omega(0)}{\omega(t)}\right]^R$$

or, from (28),

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\frac{c_1 + c_3 A^2(0)}{c_1 + c_3 A^2}\right]^{R/2} \tag{40}$$

Sophisticated mode

Write $R_1(\mu^2)$ as a series of powers of μ^2 and $\dot{\omega}/\omega$ in terms of μ – see (33)—and then integrate (39). Since

$$R_1(\mu^2) = \frac{5}{16} + \frac{29}{128}\mu^2 + \frac{359}{2048}\mu^4 + \frac{289}{2048}\mu^6 + \frac{15\,311}{131\,072}\mu^8 + \ldots \tag{41}$$

we find

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\frac{\frac{1}{2} - \mu^2(t)}{\frac{1}{2} - \mu^2(0)}\right]^{0.2348} \exp\left[G(\mu^2(t)) - G(\mu^2(0))\right] \tag{42}$$

with $G(\mu^2) = 0{\cdot}1851\mu^2 + 0{\cdot}0719\mu^4 + 0{\cdot}0374\mu^6 + 0{\cdot}0208\mu^8 + \ldots$.

Using these expressions and (29) we obtain $A(t)$.

5. Extension and improvement of CB method

We can see throughout the exposition of the above section that in the CB method it is immaterial what value the elliptic function parameter μ^2 takes. In other words, the CB method is valid (i.e. can be extended) for the three types of cubic oscillator. This means for our example that (40) and (42) are valid for all three oscillators, not just for the first. Another question that is studied below is whether the method retains its accuracy for the other oscillators and if it is possible to improve it.

Figure 1 (*a*) shows $R_1(\mu^2)$ versus the non-linear parameter v, where $R_1(\mu^2)$ has been evaluated from (36) and (37). When $0 < \mu^2 < 1$, we simply put $K \to K(\mu^2)$ and $E \to E(\mu^2)$ in these expressions. When $\mu^2 < 0$, K and E are given by (13). When $\mu^2 > 1$,

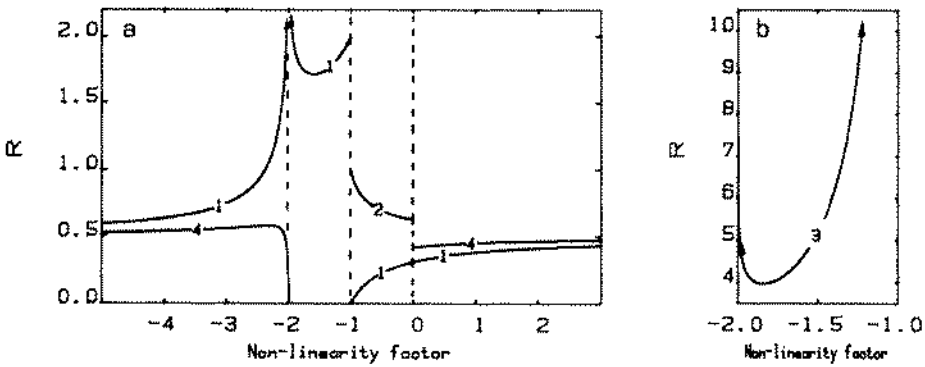
only the real part of (18) and (19) has been used, i.e.

$$\left.\begin{aligned} K &\to \eta K(\eta^2) \\ E &\to \frac{E(\eta^2) - \eta_1^2 K(\eta^2)}{\eta} \end{aligned}\right\} \tag{43}$$

This is justified if we remember that $R_1(\mu^2)$ comes from averaging elliptic functions over their real period, $T_r = 4K$, only. Notice that $Q_1(\mu^2)$ and $Q_2(\mu^2)$ are real since $\mathrm{sn}(\Psi, \mu^2)$ is real when $\mu^2 > 1$—see (15).

Figure 1 (*a*) shows that there are intervals of ν where R_1 is not approximately constant, i.e. where the simple mode of the solution is not good. Also, from (41) we see that the expansion of $R_1(\mu^2)$ in a series of powers is poor when $|\mu^2|$ is large, i.e. in such a case we should not expect the sophisticated mode to give good results.

The situation is that although the method is suitable for the three oscillators, its results are not as accurate as we want (this affirmation is corroborated in the example in § 6). This problem can be solved by means of the transformation properties of the elliptic functions. Each possibility is studied separately.



(*a*) (*b*)

Figure 1. (*a*) R versus the non-linearity factor ν; (*b*) R_3 versus ν. Line 1 gives R_1, line 2 gives R_2, and line 4 gives R_4.

5.1. *For* $\mu^2 < 0$

Here we use the negative parameter transformation. From (12) we define

$$\omega_N = \frac{\omega}{\sigma_1} \tag{44}$$

From (11)

$$\mu^2 = -\frac{\sigma^2}{1-\sigma^2}$$

and then using (33) we find

$$\frac{\dot{\omega}}{\omega} = 2(1-\mu^2)\frac{\dot{\omega}_N}{\omega_N}$$

For our example, substituting this expression into (39), we obtain

$$\frac{\dot{A}}{A} + R_2(\sigma^2)\frac{\dot{\omega}_N}{\omega_N} = -\frac{b}{2} \tag{45}$$

where

$$R_2(\sigma^2) = 2\left[1 - \left(\frac{-\sigma^2}{1-\sigma^2}\right)\right] R_1\left(-\frac{\sigma^2}{1-\sigma^2}\right) \tag{46}$$

Figure 1 (*a*) shows R_2 plotted against ν. As in the above section, (45) is solved in two modes.

Simple mode

Assuming the simplification that $R_2 \equiv R$ is constant over the integration interval, we obtain from (45)

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\frac{\omega_N(0)}{\omega_N(t)}\right]^R$$

However, from (44), $\omega_N^2 = c_1 + c_3 A^2/2$, so

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\left(c_1 + \frac{c_3 A^2(0)}{2}\right)\Big/\left(c_1 + \frac{c_3 A^2}{2}\right)\right]^{R/2} \tag{47}$$

Sophisticated mode

From (46) and using (13) we obtain

$$R_2(\sigma^2) = \frac{5}{8} + \frac{11}{64}\mu^2 + \frac{71}{1024}\mu^4 + \frac{9}{256}\mu^6 + \frac{313}{16\,384}\mu^8 + \ldots \tag{48}$$

From (44) $\dot{\omega}_N/\omega_N = -\sigma\dot{\sigma}/(1+\sigma^2)$. Then, integrating (45), we find

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\frac{1+\sigma^2(t)}{1+\sigma^2(0)}\right]^{0{\cdot}2472} \exp\left[G(\mu^2(t)) - G(\mu^2(0))\right] \tag{49}$$

with

$$G(\sigma^2) = 0{\cdot}0653\sigma^2 + 0{\cdot}0103\sigma^4 + 0{\cdot}0047\sigma^6 + 0{\cdot}0014\sigma^8 + \ldots$$

and where, from (9) and (11),

$$\sigma^2 = \left(\frac{c_3 A^2}{2}\right)\Big/\left(c_1 + \frac{c_3 A^2}{2}\right)$$

5.2. *For* $\mu^2 > 1$

Here we use the reciprocal parameter transformation. From (17) we define

$$\omega_R = \frac{\omega}{\eta} \tag{50}$$

As $\mu = 1/\eta$, from (33) we obtain

$$\frac{\dot{\omega}}{\omega} = \left(\frac{2}{\eta^2}\right)\left(\frac{\dot{\omega}_R}{\omega_R}\right)$$

Substituting this expression into (39) we obtain

$$\frac{\dot{A}}{A} + R_3(\eta^2)\frac{\dot{\omega}_R}{\omega_R} = -\frac{b}{2} \tag{51}$$

where

$$R_3(\eta^2) = \left(\frac{2}{\eta^2}\right) R_1\left(\frac{1}{\eta^2}\right) \tag{52}$$

Figure 1 (*b*) shows R_3 versus ν. Equation (51) is again solved in two modes.

Simple mode

If $R_3 \equiv R$ is constant over the integration interval then from (51):

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\frac{\omega_R(0)}{\omega_R(t)}\right]^R \tag{53}$$

From the definition (50) of ω_R, we get $\omega_R^2 = c_3 A^2/2$, and (53) then becomes

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2(1+R)}\right) \tag{54}$$

Sophisticated mode

From (52) and using (43) we obtain

$$R_3(\eta^2) = \frac{4}{\eta^2} - 1 + \frac{3}{32}\eta^2 + \frac{3}{32}\eta^4 + \frac{349}{4096}\eta^6 + \frac{314}{4096}\mu^8 + \ldots \tag{55}$$

From (50) $\dot{\omega}_R/\omega_R = \eta\dot{\eta}/(2-\eta^2)$. Substituting these expressions into (51) and integrating, we obtain

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\frac{\eta(0)}{\eta(t)}\left[\frac{2-\eta^2(t)}{2-\eta^2(0)}\right]^{0{\cdot}2472} \exp\left[G(\mu^2(t)) - G(\mu^2(0))\right] \tag{56}$$

where

$$G(\eta^2) = 0{\cdot}6177\eta^2 + 0{\cdot}1427\eta^4 + 0{\cdot}0379\eta^6 + 0{\cdot}0096\eta^8 + \ldots$$

with

$$\eta^2 = \frac{2(c_1 + c_3 A^2)}{c_3 A^2}$$

from (9) and (16).

5.3. *For* $0 < \mu^2 < 1$

In this interval we expect the expressions given by the CB method—in our example these are expressions (40) and (42)—to be good. However, it seems likely that we could increase the accuracy if the parameter changes more smoothly with respect to ν (simple mode) and/or if the parameter is smaller (sophisticated mode). This can be achieved by means of the Gauss transformation. Using (21) we define

$$\omega_G \equiv \frac{\omega}{1+\xi} \tag{57}$$

From (20)

$$\mu^2 = \frac{4\xi}{(1+\xi)^2}$$

From (57) and (33)

$$\frac{\dot{\omega}}{\omega} = \left\{\frac{4(1-\xi)}{[(1+\xi)(3-\xi)]}\right\}\left(\frac{\dot{\omega}_G}{\omega_G}\right)$$

Equation (38) then becomes

$$\frac{\dot{A}}{A} + R_4(\xi)\frac{\dot{\omega}_G}{\omega_G} = -\frac{b}{2} \tag{58}$$

where

$$R_4(\xi) = \left\{\frac{4(1-\xi)}{(1+\xi)(3-\xi)}\right\} R_1\left(\frac{4\xi}{(1+\xi)^2}\right) \tag{59}$$

Figure 1 (*a*) shows R_4 versus ν. We can see that R_4 is effectively quasi-constant over most of the range. Again our modes of resolution are given as follows.

Simple mode

Assuming $R_4 \equiv R$ is constant over the interval of integration, we obtain from (58):

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\frac{\omega_G(0)}{\omega_G(t)}\right]^R \tag{60}$$

From (57)

$$\omega_G^2 = \frac{c_1}{(1+\xi^2)-8\xi} = \frac{c_1}{(\xi-r)(\xi-s)} \tag{61}$$

with $r = 3-2\sqrt{2}$ and $s = 3+2\sqrt{2}$. Then (60) becomes

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right)\left[\frac{(\xi-r)(\xi-s)}{(\xi(0)-r)(\xi(0)-s)}\right]^{R/2} \tag{62}$$

The parameter ζ is given in terms of A by means of (20) and (9).

Sophisticated mode

From (61)

$$\frac{\dot{\omega}_G}{\omega_G} = \frac{3-\xi}{(\xi-r)(\xi-s)}\dot{\xi}$$

Defining

$$H(\xi) = R_4(\xi)\frac{3-\xi}{(\xi-r)(\xi-s)} \tag{63}$$

expression (58) becomes

$$\frac{\dot{A}}{A} + H(\xi)\dot{\xi} = -\frac{b}{2} \tag{64}$$

From (63), (59) and (22), we obtain after some algebraic calculation,

$$H(\xi) = \left(5 + \frac{9}{2}\xi - \frac{25}{8}\xi^2 + \ldots\right) \Big/ [4(\xi - r)(\xi - s)] \tag{65}$$

As ξ is very small it is sufficient to use a few terms in the numerator of (65): we only use up to first order in ξ. Notice that previously we carried out the expansion up to order eight in μ, σ, or η. However, as seen in the next section, the accuracy is not affected at all. By integrating (64) we obtain

$$\frac{A(t)}{A(0)} = \exp\left(-\frac{bt}{2}\right) \frac{[(\xi - r)/(\xi(0) - r)]^{\alpha_2}}{[(\xi - s)/(\xi(0) - s)]^{\alpha_1}} \tag{66}$$

with $\alpha_1 = 1{\cdot}3801$ and $\alpha_2 = 0{\cdot}2551$.

Finally it is interesting to note that Bravo Yuste and Díaz Bejarano (1987) have developed a method designed exclusively for solving our example oscillator, obtaining expressions for $A(t)$ very similar to (47), (49) and (56), but a little more accurate. Equation (54) is the same as their equation (58).

6. Example

In this section each technique of solution is applied to some illustrative cases of the example oscillator (6). The results are given in Tables 1–6, which we now discuss in detail. The second column gives the maximum values of $|x(t)|$ evaluated by numerical integration using a fourth-order Runge–Kutta method. Only a certain number of significant values are given.

t	$A(t)$	A_u	A_r	A_s^1	A_s^2	A_{si}	A_{sp}	A_{spi}
2·433	0·902	0·885	0·907	0·906	0·901	0·904	0·902	0·902
4·965	0·808	0·780	0·820	0·817	0·807	0·811	0·808	0·809
7·591	0·720	0·684	0·738	0·731	0·718	0·724	0·720	0·720
10·304	0·637	0·597	0·662	0·650	0·636	0·642	0·637	0·637
27·910	0·276	0·248	0·327	0·289	0·276	0·280	0·276	0·276

Table 1. (Hard oscillator.) For $c_1 = 1$, $c_3 = 1$, $b = 0{\cdot}1$, $x(0) = 1$, $\dot{x}(0) = 0$: $A(t)$ is the maximum value of $|x(t)|$ evaluated numerically; A_u is the amplitude evaluated using the unrefined approximation; A_r is the amplitude using the refined approximation; A_s^1 is the amplitude using the simple mode with $R_1 = 1/2$; A_s^2 is the same as A_s^1 but with $R_1 = 0{\cdot}35$; A_{si} is the amplitude given by the improved simple mode; A_{sp} is the amplitude given by the sophisticated mode; and A_{spi} is the same as A_{sp} but using the improved sophisticated mode.

t	$A(t)$	A_u	A_r	A_s^1	A_s^2	A_{si}	A_{sp}	A_{spi}
1·898	4·603	4·948	4·720	4·725	4·649	4·601	4·653	4·555
3·496	4·713	5·359	4·914	4·930	4·782	4·722	4·799	4·623
4·920	4·828	5·755	5·093	5·126	4·908	4·850	4·942	4·702
10·623	5·434	7·654	5·878	6·038	5·478	5·525	5·652	5·171
20·005	6·948	12·235	7·443	8·048	6·670	7·161	7·309	6·467

Table 2. (Soft–hard oscillator with $E > 0$.) For $c_1 = -10$, $c_3 = 1$, $b = -0{\cdot}1$, $x(0) = 4{\cdot}5$, $\dot{x}(0) = 0$: the key is the same as that in Table 1, but now using $R_1 = 1$ for A_s^2.

We begin with two cases of $0 < \mu^2 < 1$. The first (see Table 1) is a hard oscillator ($0 < \mu^2 < 1/2$) with $c_1 = 1$, $c_3 = 1$, $b = 0{\cdot}1$, $x(0) = 1$, and $\dot{x}(0) = 0$. We see that the oscillations are mildly non-linear, i.e. $v(t = 0) = 1$ and $v(t = 27{\cdot}91) = 0{\cdot}076$. As shown in § 3, the amplitude given by the unrefined approximation A_u is the same as that of a linear oscillator with equal damping. For this reason A_u is good and it is even better as the oscillations become more linear, i.e. progressively smaller. In the refined approximation, the amplitude A_r decays as in the unrefined one, but with the damping

t	$A(t)$	A_u	A_r	A_s	A_{si}	A_{sp}	A_{spi}
1·032	0·974	0·974	0·973	0·974	0·974	0·974	0·974
2·062	0·948	0·950	0·947	0·948	0·948	0·948	0·948
3·090	0·923	0·926	0·922	0·923	0·923	0·923	0·923
4·115	0·899	0·902	0·897	0·899	0·899	0·899	0·899
5·140	0·876	0·880	0·874	0·876	0·876	0·876	0·876
11·255	0·749	0·755	0·744	0·749	0·749	0·749	0·749
20·351	0·595	0·601	0·585	0·595	0·595	0·595	0·595

Table 3. (Hard–soft oscillator.) For $c_1 = 10$, $c_3 = -1$, $b = 0{\cdot}05$, $x(0) = 1$, $\dot{x}(0) = 0$: the key is the same as that in Table 1, but now A_s is evaluated using $R_1 = 5/16$ and A_{si} using $R_2 = 5/8$.

t	$A(t)$	A_u	A_s^1	A_s^2	A_{si}^1	A_{si}^2	A_{spi}
1·756	2·726	2·871	2·231	2·395	2·771	2·751	2·720
3·242	2·562	2·766	2·119	2·271	2·616	2·589	2·557
4·630	2·434	2·672	2·025	2·170	2·489	2·459	2·429
5·956	2·324	2·585	1·942	2·081	2·380	2·347	2·320
7·238	2·228	2·503	1·868	2·001	2·282	2·249	2·224
10·908	1·988	2·284	1·678	1·798	2·038	2·004	1·985
19·990	1·536	1·820	1·308	1·403	1·574	1·543	1·533
35·878	1·008	1·223	0·864	0·928	1·032	1·011	1·007

Table 4. (Hard–soft oscillator.) For $c_1 = 10$, $c_3 = -1$, $b = 0{\cdot}05$, $x(0) = 3$, $\dot{x}(0) = 0$: the key is the same as that in Table 1, but now A_s^1 is evaluated using $R_1 = 5/16$, A_s^2 using $R_1 = 0{\cdot}25$, A_{si}^1 using $R_2 = 5/8$, and A_{si}^2 using $R_2 = 0{\cdot}7$.

t	$A(t)$	A_u	A_r	A_s	A_{si}	A_{sp}	A_{spi}
0·959	0·907	1·244	1·045	0·918	0·908	0·959	0·954
1·926	4·358	4·193	4·297	4·354	4·358	4·337	4·338
2·836	1·085	1·789	1·394	1·109	1·087	1·198	1·193
3·752	4·319	4·006	4·202	4·311	4·318	4·281	4·282
4·628	1·230	2·154	1·650	1·260	1·231	1·373	1·376
5·510	4·281	3·834	4·112	4·271	4·280	4·232	4·228
10·525	4·175	3·382	3·866	4·163	4·174	4·108	4·085
16·852	4·051	2·887	3·577	4·038	4·044	3·986	3·921

Table 5. (Soft–hard oscillator with $E < 0$.) For $c_1 = -10$, $c_3 = 1$, $b = 0{\cdot}05$, $x(0) = 4{\cdot}4$, $\dot{x}(0) = 0$: $A(t)$ denotes the maximum value and also the minimum value of $|x(t)|$. The key is the same as that in Table 1, but now A_s is evaluated using $R_1 = 1{\cdot}7$ and A_{si} using $R_3 = 4$.

t	$A(t)$	A_u	A_r	A_s	A_{si}^1	A_{si}^2	A_{sp}	A_{spi}
0·712	2·673	2·816	2·707	2·673	2·688	2·672	2·659	2·684
1·428	3·572	3·352	3·520	3·570	3·549	3·572	3·592	3·555
2·138	2·709	3·088	2·807	2·711	2·754	2·710	2·669	2·741
2·852	3·545	3·122	3·443	3·543	3·499	3·543	3·585	3·513
3·561	2·742	—	2·900	2·746	2·816	2·746	2·679	2·791
5·693	3·497	2·708	3·293	3·492	3·401	3·488	3·571	3·441
12·766	3·401	1·902	2·947	3·391	3·169	3·354	3·543	3·314
16·997	3·357	1·539	2·758	3·345	3·037	3·276	3·529	3·265
24·038	3·300	1·082	2·470	3·287	2·831	3·150	3·510	3·215

Table 6. (Soft–hard oscillator with $E < 0$.) For $c_1 = -10$, $c_3 = 1$, $b = 0{\cdot}1$, $x(0) = 3{\cdot}6$, $\dot{x}(0) = 0$: the key is the same as that in Table 1, but now A_{si}^1 is evaluated using $R_3 = 4$ and A_{si}^2 using $R_3 = 8$.

coefficient modified by the factor $(1 + \mu^2(0))^{-1}$. For low t this correction is fortunate and A_r is good. Moreover, for large t the oscillations are quasi-linear and this factor should be near unity. However, the factor in this method is constant and therefore A_r progressively worsens. The amplitude A_s evaluated by (40) (simple mode) with $R_1 = 1/2$ is quite good and better than the above ones. The results are given in the column labeled A_s^1. From Fig. 1 (*a*) we see that for these very linear regions, with v so small, R_1 must be chosen smaller. Thus, if we use $R_1 = 0{\cdot}35$, the approximation as given in the column labeled A_s^2 is very good. The amplitude A_{si} given by (62) (improved simple mode) with $R_4 = 0{\cdot}5$ is also very good; of course a better choice of R_4 (e.g. $R_4 = 0{\cdot}43$) will give better results. Finally, amplitudes evaluated by the sophisticated mode A_{sp}—(42)—and by the improved sophisticated mode A_{spi}—(66)—are the best. This is not surprising since μ^2 and ξ are small and hence the series expansions of $R_1(\mu^2)$ and $H(\xi)$ are very accurate.

The second case is a soft–hard oscillator with negative damping and with oscillations near the interesting zone $E = 0$: $c_1 = 10$, $c_3 = 1$, $b = -0{\cdot}1$, $x(0) = 4{\cdot}5$, and $\dot{x}(0) = 0$ (see Table 2). Then $v(t) < -2$ and $1/2 < \mu^2 < 1$. The oscillations are very non-linear, which is why A_u is bad. However, A_r is a lot better. The choice of $R_1 = 1/2$ for evaluating A_s is not good—see Fig. 1 (*a*). It would be better to take $R_1 = 1$ (see Table 2). Although the choice of $R_4 = 0{\cdot}5$ in evaluating A_{si} is now rather small—see Fig. 1 (*a*)—the improved simple mode nevertheless gives an excellent approximation. We now see that A_{sp} is not as good because μ^2 is large. The same happens with A_{spi} and its parameter ξ.

Next we show two cases of the oscillator 2 (hard–soft), i.e. with $\mu^2 < 0$. In the first $c_1 = 10$, $c_3 = -1$, $b = 0{\cdot}05$, $x(0) = 1$, and $\dot{x}(0) = 0$ (see Table 3). As the initial amplitude is small, the oscillations are very linear: $v(0) = -0{\cdot}1$. We see in Table 3 that all of the approximations are good, especially that of the CB method. In the simple mode approximations, R_1 and R_2—(60)—are nearly constant (and equal to 5/16 and 5/8, respectively) and in the sophisticated mode, $|\mu^2|$ and σ^2—(49)—are very small, which is why they are so accurate.

In the second case $c_1 = 10$, $c_3 = -1$, $b = 0{\cdot}05$, $x(0) = 3$, and $\dot{x}(0) = 0$ (see Table 4). For low t the oscillations are very non-linear, $v(0) \simeq -1$, the parameter is very large ($|\mu^2| \simeq 4$), and it changes quickly with time. For this, A_r, A_{si}, and A_{sp} are not good. In fact, A_r and A_{sp} are so bad that their values are not significant and they are not shown in the table. Figure 1 (*a*) shows that in this region R_1 and R_2 are not quasi-constant,

and consequently A_s and A_{si} are not good (however a good choice of R_1 or R_2—e.g. $R_1 = 0{\cdot}25$, $R_2 = 0{\cdot}7$—still gives results that are not so bad). On the other hand, the approximation of the improved sophisticated mode is very good since σ^2 is small.

Finally, let us discuss two cases of oscillator 3 (soft–hard) with $E < 0$, i.e. with $\mu^2 > 1$. Observe now that the oscillations are not symmetrical. Of course all oscillations have maximum elongations A and minimum A_{low}. However, for symmetrical oscillations $|A_{\text{low}}| = A$, and therefore we have not discussed A_{low} before. However, now $|A_{\text{low}}| \neq A$, and in fact $A_{\text{low}} = \eta_1 A$ since the minimum of cn with $\mu^2 > 1$, i.e. the minimum of dn, is η_1. So, for these two cases, we show A and A_{low} alternatively (although, for brevity, only for the first values).

In the first case $c_1 = -10$, $c_3 = 1$, $b = 0{\cdot}05$, $x(0) = 4{\cdot}4$, and $\dot{x}(0) = 0$ (see Table 5). We see that A_u and A_r are bad, although A_r is better. A_s with $R_1 = 1{\cdot}7$—see Fig. 1 (*a*)—and especially A_{si}—(54)—with $R_3 = 4$—see Fig. 1 (*b*)—are much better. The values of A_{sp} and A_{spi}—(56)—are similar and good, but not as good as A_{si}. It is not strange that A_{sp} is good since $\mu^2(t)$ is not very large, e.g. for $t = 16{\cdot}85$, $\mu^2 = 1{\cdot}28$.

The second case is $c_1 = -10$, $c_3 = 1$, $b = 0{\cdot}1$, $x(0) = 3{\cdot}6$, and $\dot{x}(0) = 0$ (see Table 6). Here A_u and A_r are very bad, especially A_u. In fact, A_u is so bad that for $t = 3{\cdot}561$—and, of course, for larger t—A_{low} is absurd: A is too small and so η_1 is imaginary. As the oscillations are closer to the bottom of the potential well than in the above case (now $v(0) = -1{\cdot}296$, where before $v(0) = -1{\cdot}936$) μ^2 is larger and changes more quickly, and η^2 is smaller than before. This is the reason that A_{sp} is now worse, whereas A_{spi} is better. Finally, from Fig. 1 (*a*) one sees that $R_1 = 1{\cdot}7$ must be a good value for evaluating A_s. In fact this approximation is better than the approximation of the improved simple method since R_3 changes very much in this region of oscillation.

7. Conclusions

(i) The KB elliptic methods examined here are more general than has generally been thought, since they are directly applicable not only to oscillator 1 (hard), but also to oscillator 2 (hard–soft) and oscillator 3 (soft–hard). Moreover, as has been shown, in many cases they give very good results, especially the CB method.

(ii) We have studied in detail the more precise KB elliptic method—the CB one—and we have shown that in many cases it is possible to improve it using the transformation properties with respect to the parameter of the elliptic functions. For the example studied, very accurate (and some very simple) expressions for the amplitude oscillations have been obtained.

Finally, we point out that the extension presented here enables one to apply the technique of Soudack (1974) to $\ddot{x} + g(x) + \varepsilon f(x, \dot{x}, t) = 0$, with $g(x)$ odd, without the restriction that the first two coefficients of the power expansion proposed for $g(x)$ be positive.

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