Journal of Sound and Vibration (1987) **114**(1), 33-44

AMPLITUDE DECAY OF DAMPED NON-LINEAR OSCILLATORS STUDIED WITH JACOBIAN ELLIPTIC FUNCTIONS

S. BRAVO YUSTE AND J. DIAZ BEJARANO

Departamento de Termodinámica, Universidad de Extremadura, 06071-*Badajoz, Spain*

(*Received* 23 *January* 1986)

Approximate bounded solutions of the equation $x''+bx'+c_1x+c_3x^3=0$ with $b>0$, $c_1 \gtrless 0$ and $c_3 \gtrless 0$ are developed in terms of the Jacobian elliptic functions cn, cd and dn. The solutions are found by following the method developed by Christopher in 1973 for the case with $c_1>0$ and $c_3>0$. Formulas for the amplitude decay are given in two different approximations. The solutions are compared with Runge-Kutta numerical integration results and shown to be accurate for a wide range of b, c_1, c_3, and initial conditions.

1. INTRODUCTION

Damped non-linear oscillators governed by

$$x''+bx'+c_1x+c_3x^3=0 \tag{1}$$

or, equivalently

$$(x')^2+V(x)=E(t), \tag{2}$$

where $V(x)=c_1x^2+(c_3/2)x^4$, have long been a subject of interest and study (Burton [1] has given a survey of these efforts). However, most results have been obtained for $c_1>0$ and $c_3>0$ (well 1). We have used the approximate Christopher method [2, 3] (similar to that of Krylov and Bogoliubov [4] but using elliptic functions rather than circular functions as a basis for the solution) and studied the other oscillators for which

$$c_1>0, \qquad c_3<0 \quad \text{where} \quad E(0)<V_{\max}=-c_1^2/2c_3 \text{ (well 2)} \tag{3}$$

and $c_1<0$, $c_3>0$ (well 3).

In this paper our attention is focused on the amplitude decay: $A(t)$. From knowledge of $A(t)$ it is not difficult to evaluate other magnitudes, such as the energy, or $x(t)$, as done in references [1-3].

The Christopher method consists of transforming equation (1) to the reduced equation

$$u''+m^2u+m^2\gamma(t)u^3=0, \tag{4}$$

and using the transformation

$$x(t)=x(0)\exp(-kt)u(t), \tag{5}$$

where

$$k=b/2, \qquad m^2=c_1-k^2, \qquad m^2\gamma(t)=c_3x^2(0)\exp(-2kt). \tag{6}$$

(When possible we use the Christopher notation in this paper.)

0022-460X/87/010033+12 $03.00/0

Equation (2) is also equivalent to

$$(u')^2 + V_1(u) = E_1(t), \tag{7}$$

where

$$V_1(u) = m^2u^2 + \{m^2\gamma(t)/2\}u^4. \tag{8}$$

The case $m^2 > 0$ and $m^2\gamma(t) > 0$ we call oscillator 1; $m^2 > 0$, $m^2\gamma(t) < 0$ and $E_1(0) < V_1(0) = -(c_1 - k^2)^2/2c_3x(0)^2$, we call oscillator 2; and $m^2 < 0$ and $m^2\gamma(t) > 0$ we call oscillator 3.

If $\gamma(t)$ in equations (4) or (8) were constant and not time-dependent as it is, the exact solution (generating solution) of this differential equation (generating equation), would be given in terms of Jacobian elliptic functions as

$$u(t) = c\ \mathrm{pq}\ (\omega t - \phi, \mu^2), \tag{9}$$

where c, ω, ϕ, and μ are constants and pq is one of the Jacobian elliptic functions depending on the sign of m^2 and $m^2\gamma(t)$ (see Table 1). The next step is the usual one in the Krylov–Bogoliubov method. One assumes that the solution to equation (4), with $\gamma(t)$ time-dependent, has the form of equation (9) where c, ω, ϕ, μ are not constants, but time-dependent. That is, one supposes the solution to be

$$u(t) = c(t)\ \mathrm{pq}\ (\Psi(t), \mu^2(t)), \tag{10}$$

where

$$\Psi(t) = \int_0^t \omega(t)\, \mathrm{d}t - \phi(t). \tag{11}$$

TABLE 1

Formulas for the oscillators $(x')^2 + Ax^2 + Bx^4 = E$; *here* $E_R = E/V_m$ *where* $V_m = A^2/4B$

Differential equation	Solution x	ω^2	μ^2	$\mu^2(E_R)$
$A>0$ $B>0$ $E>0$	$c\ \mathrm{cn}\ (\omega t, \mu^2)$	$A+2Bc^2$ $(A^2+4BE)^{1/2}$ $\omega^2 > A$	$Bc^2/(A+2Bc^2)$ $0<\mu^2<\frac{1}{2}$	$\frac{1}{2}\left\{1 - \frac{(1-E_R)^{1/2}}{1-E_R}\right\}$
$A>0$ $B<0$ $0<E<V_m$	$c\ \mathrm{cd}\ (\omega t, \mu^2)$	$A+Bc^2$ $\frac{1}{2}(A+(A^2+(BE)^{1/2})$ $A/2<\omega^2<A$	$Bc^2/(A+Bc^2)$ $0<\mu^2<1$	$\frac{2-E_R}{E_R} - \frac{2(1-E_R)^{1/2}}{E_R}$
$A<0$ $B>0$ $E>0$	$c\ \mathrm{cn}\ (\omega t, \mu^2)$	$A+2Bc^2$ $(A^2+4BE)^{1/2}$ $\omega^2 > A$	$Bc^2/(A+2Bc^2)$ $\frac{1}{2}<\mu^2<1$	$\frac{1}{2}\left\{1 + \frac{(1-E_R)^{1/2}}{1-E_R}\right\}$
$A<0$ $B>0$ $V_m<E<0$ $E>0$	$c\ \mathrm{dn}\ (\omega t, \mu^2)$	Bc^2 $\frac{1}{2}(-A+(A^2+(BE)^{1/2})$ $A/2<\omega^2<-A$ $\omega^2>-A$	$2+(A/Bc^2)$ $0<\mu^2<1$ $1<\mu^2<2$	$\frac{2E_R-2}{E_R} + \frac{2(1-E_R)^{1/2}}{E_R}$

2. CONSTRAINTS

The functions $c(t)$ (reduced amplitude), $\omega(t)$ (frequency), $\phi(t)$ (phase) and $\mu^2(t)$ (parameter) can be anything, subject to the following obvious constraint: constraint 1: equation (10) must be a solution of equation (4).

We have found it practical to impose three additional constraints on all three oscillators: constraint 2: if $f(c, \omega, \phi, \mu)$ is the time derivative of equation (9) then the time derivative of equation (10) must be

$$u'(t) = f\{c(t), \omega(t), \phi(t), \mu(t)\}; \tag{12}$$

constraint 3:

$$\omega^2 = g\{m^2, m^2\gamma(t), c(t)\}; \tag{13}$$

constraint 4:

$$\mu^2 = h\{m^2, m^2\gamma(t), c(t)\}. \tag{14}$$

Here g and h are the relations (shown in Table 1) between frequency, parameter, and reduced amplitude of the generating solution with coefficients m^2 and $m^2\gamma(t)$ of the generating equation. The task of obtaining the solution $u(t)$ is thus transformed into one of obtaining the functions $c(t)$, $\omega(t)$, $\phi(t)$ and $\mu(t)$ that satisfy the four constraints. In what follows here the details of this system are given for one oscillator, oscillator 2 (for the first one see reference [2], and, for all three, see the Master's Thesis (Licenciatura) of one of the authors (S.B.Y.) [5]). For oscillator 2 the assumed solution is

$$u(t) = c(t)\,\mathrm{cd}\,(\Psi(t), \mu^2(t)), \tag{15}$$

where $\Psi(t)$ is given by equation (11). Differentiating equation (15) with respect to t gives

$$u'(t) = -c\omega\mu_1^2\,\mathrm{sd}\,\mathrm{nd} + c'\,\mathrm{cd} + c\phi'\mu_1^2\,\mathrm{sd}\,\mathrm{nd} + (cI\mu'\,\mathrm{sd}\,\mathrm{nd})/\mu, \tag{16}$$

where

$$I = \mathrm{E}(\Psi, \mu) - \mu_1^2\Psi, \qquad \mu_1^2 = 1 - \mu^2, \tag{17}$$

and $\mathrm{E}(\Psi, \mu)$ is the incomplete elliptic integral of the second kind. From constraint 2,

$$u'(t) = -c\omega\mu_1^2\,\mathrm{sd}\,\mathrm{nd}. \tag{18}$$

Then from equation (16)

$$c'\,\mathrm{cd} + c\phi'\mu_1^2\,\mathrm{sd}\,\mathrm{nd} + (cI\mu'\,\mathrm{sd}\,\mathrm{nd})/\mu = 0. \tag{19}$$

A second differentiation of equation (18) gives

$$\begin{aligned} u'' = &-c\omega\mu_1^2\,\mathrm{sd}\,\mathrm{nd} - c\omega'\mu_1^2\,\mathrm{sd}\,\mathrm{nd} - 2c\omega\mu_1\mu_1'\,\mathrm{sd}\,\mathrm{nd} - \{\omega^2 c\mu_1^2\,\mathrm{cd}\,(\mathrm{nd}^2 + \mu^2\,\mathrm{sd}^2) \\ &- c\omega\mu_1^2\phi'\,\mathrm{cd}\,(\mathrm{nd}^2 + \mu^2\,\mathrm{sd}^2) + \omega c(\mu'/\mu)\,\mathrm{cd}\,\mathrm{nd}^2\,(\mu^2\,\mathrm{dn}\,\mathrm{sc} - I) \\ &+ \omega c\mu'\mu\,\mathrm{cd}\,\mathrm{sd}^2\,(\mathrm{dn}\,\mathrm{sc} - I)\}. \end{aligned} \tag{20}$$

From constraint 3

$$\omega^2 = m^2(1 + \tfrac{1}{2}\gamma c^2) \tag{21}$$

and from constraint 4

$$\mu^2 = (-\tfrac{1}{2}\gamma c^2)/(1 + \tfrac{1}{2}\gamma c^2). \tag{22}$$

Then there follows immediately

$$m^2 = (1 + \mu^2)\omega^2, \qquad m^2\gamma = -2\omega^2\mu^2/c^2. \tag{23a,b}$$

With these relations and with (see reference [6])

$$\mathrm{cd}^2 = 1 - \mu_1^2\,\mathrm{sd}^2, \qquad \mu^2\,\mathrm{cd}^2 = \mu^2 - \mu_1^2\,\mathrm{nd}^2 \tag{24}$$

one obtains

$$m^2 u + m^2\gamma u^3 = \omega^2 c\mu_1^2\,\mathrm{cd}\,(\mathrm{nd}^2 + \mu^2\,\mathrm{sd}^2). \tag{25}$$

From the constraint 1, and upon using equations (20) and (25), one has

$$-c'\omega\mu_1^2\,\mathrm{sd}\,\mathrm{nd} - c\omega'\mu_1^2\,\mathrm{sd}\,\mathrm{nd} - 2c\omega\mu_1\mu_1'\,\mathrm{sd}\,\mathrm{nd} + \omega c\mu_1^2\phi'\,\mathrm{cd}\,(\mathrm{nd}^2 + \mu^2\,\mathrm{sd}^2)$$
$$-\omega c(\mu'/\mu)\,\mathrm{cd}\,\mathrm{nd}^2(\mu^2\,\mathrm{dn}\,\mathrm{sc} - I) - \omega c\mu'\mu(\mathrm{dn}\,\mathrm{sc} - I)\,\mathrm{cd}\,\mathrm{sd}^2 = 0. \tag{26}$$

The system to solve is that of equations (19), (21), (22) and (26). For the other oscillators the system has a similar complexity.

The initial conditions to be satisfied by the solutions of oscillator 2 are

$$\S_1 = u(0) = 1 = c(0)\,\mathrm{cd}\,(-\phi(0), \mu^2(0)),$$
$$\S_2 = u'(0) = [x'(0) + kx(0)]/x(0)$$
$$= -c(0)\omega(0)\mu_1^2(0)\,\mathrm{sd}\,(-\phi(0), \mu^2(0))\,\mathrm{nd}\,(-\phi(0), \mu^2(0)). \tag{27}$$

Then one can deduce that

$$c^2(0) = -\{m^2 + (m^4 + 2m^2\gamma E_1(0))^{1/2}\}/m^2\gamma, \tag{28}$$

where

$$E_1(0) = \S_1^2 + m^2\S_2^2 + \{m^2\gamma(0)/2\}\S_2^4. \tag{29}$$

For the other oscillators $c^2(0)$ is given by the same expression.

3. FUNDAMENTAL RELATIONS

Following Christopher [2], we first derive an equation for each oscillator that we shall call the fundamental relationship, and we shall then apply the principle of averaging to this. Let us follow the process through for oscillator 2. From equation (19) we have

$$c\mu_1^2\phi' = -(cI(\mu'/\mu)\,\mathrm{sd}\,\mathrm{nd} + c'\,\mathrm{cd})/(\mathrm{sd}\,\mathrm{nd}). \tag{30}$$

Then constraints 3 and 4 give

$$\mu\mu' = -(1 + \mu^2)\omega'/\omega. \tag{31}$$

Using equations (26), (30) and (31), and writing all the Jacobian elliptic functions in terms of sd as in equation (24), one gets the fundamental relationship

$$(c'/c)(1 + 2\mu^2\,\mathrm{sd}^2 - \mu^2\mu_1^2\,\mathrm{sd}^4) + (\omega'/\omega)(2\,\mathrm{sd}^2 - \mu_1^2\,\mathrm{sd}^4) = 0. \tag{32}$$

For the other oscillators the fundamental relation is obtained in a similar way. So, for oscillator 1, one has [2]

$$(c'/c)(1 - 2\mu^2\,\mathrm{sn}^2 + \mu^2\,\mathrm{sn}^4) + (\omega'/\omega)(\mathrm{sn}^2 - \tfrac{1}{2}\,\mathrm{sn}^4) = 0 \tag{33}$$

and, for oscillator 3 [5],

$$(c'/c)(1 - 2\,\mathrm{sn}^2 + \mu^2\,\mathrm{sn}^4) + (\omega'/\omega)(2\,\mathrm{sn}^2 - \mu^2\,\mathrm{sn}^4) = 0. \tag{34}$$

There is however a more elegant way: if we know one of the fundamental relations we can obtain the other two using the transformation properties of the Jacobian elliptic functions [6]: transformation of the negative parameter, transformation of the reciprocal parameter (Gauss's transformation) and Jacobi's imaginary transformation. For example, one can deduce (32) from equation (33). One knows that the solution to equation (7) for $m^2>0$, $m^2\gamma(t)<0$ and $E_1(0)<V_{1max}(0)$ has the form of equation (15) with $\mathrm{pq}\equiv\mathrm{cd}$. But one also knows that (the negative parameter transformation)

$$\mathrm{cn}\,(\Psi_{\mathrm{cn}},-\mu^2_{\mathrm{cn}})=\mathrm{cd}\,(\Psi_{\mathrm{cd}},\mu^2_{\mathrm{cd}}),\tag{35}$$

where

$$\mu^2_{\mathrm{cd}}=\mu^2_{\mathrm{cn}}/(1+\mu^2_{\mathrm{cn}}),\qquad \Psi_{\mathrm{cd}}=(1+\mu^2_{\mathrm{cn}})^{1/2}\Psi_{\mathrm{cn}}=(1-\mu^2_{\mathrm{cd}})^{-1/2}\Psi_{\mathrm{cn}}.\tag{36, 37}$$

In other words one can take the solution for oscillator 2 to be

$$c\,\mathrm{cn}\,(\Psi_{\mathrm{cn}},-\mu^2_{\mathrm{cn}}),\tag{38}$$

where, from equation (11),

$$\Psi_{\mathrm{cn}}=\int_0^t\omega_{\mathrm{cn}}\,\mathrm{d}t-\phi_{\mathrm{cn}}(t).\tag{39}$$

But cn is the Jacobian elliptic function used for oscillator 1, and in deducing the fundamental relation, no use has been made of the values of the argument or parameter. Thus equation (33) would also be the fundamental relation for oscillator 2, with a negative parameter. One can now check that this is indeed the same as equation (32). Reference [6] gives the relation

$$\mathrm{sn}\,(\Psi_{\mathrm{cn}},-\mu^2_{\mathrm{cn}})=(1-\mu^2_{\mathrm{cd}})^{1/2}\,\mathrm{sd}\,(\Psi_{\mathrm{cd}},\mu^2_{\mathrm{cd}}).\tag{40}$$

From equation (37),

$$\omega_{\mathrm{cn}}=(1-\mu^2_{\mathrm{cd}})^{1/2}\omega_{\mathrm{cd}}\tag{41}$$

and therefore

$$(\omega'_{\mathrm{cn}}/\omega_{\mathrm{cn}})=(\omega'_{\mathrm{cd}}/\omega_{\mathrm{cd}})-(\mu'_{\mathrm{cd}}\mu_{\mathrm{cd}}/1-\mu^2_{\mathrm{cd}}).\tag{42}$$

Using equation (31) gives

$$\omega'_{\mathrm{cn}}/\omega_{\mathrm{cn}}=(2/(1-\mu^2_{\mathrm{cd}}))\omega'_{\mathrm{cd}}/\omega_{\mathrm{cd}}.\tag{43}$$

Upon substituting this last equation and equation (40) into equation (33), it is easy to see that equation (32) follows: q.e.d.

4. PRACTICAL SOLUTION

Although the fundamental relations are far simpler than the system they derive from, they are still complex enough for their exact resolution to be difficult. However, the coefficients of ω'/ω and c'/c are roughly periodic and one can reduce equations (32)–(34) to a simpler form by using the averaging principle method [4]. The solutions of these averaged relations are closer to the solutions of the fundamental relations when the Jacobian elliptic functions' frequency $\Psi'=\omega-\phi'$ is large and c', ω', μ' and ϕ' are small (that is, when the damping is small). In Table 1 one can see that for oscillators 1 and 3 (with $E_1>0$) the frequency can be very high for very large energy, and then the technique will also be valid for large b. But for oscillator 2 the frequency is very small ($(c_1-k^2)/2<\omega^2<c_1-k^2$), and the technique works well only when b is small. The same is the case for the oscillator 3 when $E_1<0$.

Averaging the fundamental relations gives

$$(c'/c)+R(\mu^2)(\omega'/\omega)=0,\tag{44}$$

where

$$R(\mu^2) = (Q_1 + \tfrac{1}{2}Q_2)/(1 - 2\mu^2 Q_1 + \mu^2 Q_2) \tag{45}$$

for oscillator 1,

$$R(\mu^2) = (2T_1 - \mu_1^2 T_2)/(1 + 2\mu^2 T_1 - \mu^2\mu_1^2 T_2) \tag{46}$$

for oscillator 2, and

$$R(\mu^2) = (2Q_1 - \mu^2 Q_2)/(1 - 2Q_1 + \mu^2 Q_2) \tag{47}$$

for oscillator 3. In this oscillator, if $E_1 > 0$, then $1 < \mu^2 < 2$. If one wants to use the Jacobian elliptic functions with parameters between zero and one, a reciprocal parameter transformation can be applied to equation (34) and, after averaging, this gives

$$R(\mu^2) = \{2Q_1(1/\mu^2) - Q_2(1/\mu^2)\}/\{\mu^2 - 2Q_1(1/\mu^2) + Q_2(1/\mu^2)\} \tag{48}$$

for oscillator 3 when $E_1 > 0$. Here the notation

$$Q_1(\mu^2) \equiv \langle \mathrm{sn}^2 \rangle = \{1 - [\mathrm{E}(\mu^2)/\mathrm{K}(\mu^2)]\}/\mu^2, \tag{49}$$

$$Q_2(\mu^2) \equiv \langle \mathrm{sn}^4 \rangle = \{2 + \mu^2 - 2(1+\mu^2)(\mathrm{E}/\mathrm{K})\}/(3\mu^4), \tag{50}$$

$$T_1(\mu^2) \equiv \langle \mathrm{sd}^2 \rangle = \{(\mathrm{E}/\mathrm{K}) - \mu_1^2\}/(\mu^2\mu_1^2), \tag{51}$$

$$T_2(\mu^2) \equiv \langle \mathrm{sd}^4 \rangle = \{2(2\mu^2 - 1)(\mathrm{E}/\mathrm{K}) + \mu_1^2(2 - 3\mu^2)\}/(3\mu^4\mu_1^4), \tag{52}$$

where

$$\langle \mathrm{pq}\,(\Psi, \mu^2) \rangle = \frac{1}{4\mathrm{K}(\mu^2)} \int_0^{4\mathrm{K}(\mu^2)} \mathrm{pq}\,(\Psi, \mu^2)\,\mathrm{d}\Psi \tag{53}$$

and $\mathrm{E} \equiv \mathrm{E}(\mu^2)$ and $\mathrm{K} \equiv \mathrm{K}(\mu^2)$ are the complete elliptic integrals of the second and first kind, has been used. We shall show below how to get information from equation (44) about the amplitude decay. It should then be possible to evaluate $\Psi(t)$, $\mu(t)$, and $\omega(t)$, and to obtain the solution $x(t)$. However, the present paper is restricted to the study of the amplitude decay. This study will be carried out in two ways, that we shall call the simple and sophisticated modes, respectively.

As oscillator 3 is not symmetric for $E_1 < 0$, there are two amplitude decays: the upper or $A(t)$ (corresponding to the maximum elongation) and lower $A_{low}(t)$ (minimum elongation point). But the minimum value of $\mathrm{d}n(\Psi, \mu^2)$ is μ_1 and so

$$A_{low}(t) = \mu_1(t)A(t). \tag{54}$$

In evaluating Table 4 (to follow) oscillator 3 ($E_1 > 0$), we took the solution to be of type cn ($\frac{1}{2} < \mu^2 < 1$) (see Table 1): that is, we used the relations of oscillator 1.

4.1. SIMPLE MODE RESULTS

The basis of the simple mode approach is to assume that $R(\mu^2) \equiv R$ is time-independent. Integration of equation (44) is then simple:

$$c(t)/c(0) = \{\omega(0)/\omega(t)\}^R. \tag{55}$$

Using constraint 3 (see Table 1) one gets

$$\frac{c(t)}{c(0)} = \left\{\frac{1 + \gamma(0)c^2(0)}{1 + \gamma(t)c^2(t)}\right\}^{R/2}, \quad \text{or} \quad A(t) = A(0)\exp(-kt)\left\{\frac{1 + \gamma(0)A^2(0)}{1 + \gamma(0)A^2(t)}\right\}^{R/2}, \tag{56}$$

for oscillator 1 (and for oscillator 3 when $E_1 > 0$, too),

$$\frac{c(t)}{c(0)} = \left\{\frac{1+\frac{1}{2}\gamma(0)c^2(0)}{1+\frac{1}{2}\gamma(t)c^2(t)}\right\}^{R/2} \quad \text{or} \quad A(t) = A(0)\exp(-kt)\left\{\frac{1+\frac{1}{2}\gamma(0)A^2(0)}{1+\frac{1}{2}\gamma(0)A^2(t)}\right\}^{R/2} \tag{57}$$

for oscillator 2, and

$$\frac{c(t)}{c(0)} = \exp\left(\frac{R}{1+R}kt\right), \quad \text{or} \quad A(t) = A(0)\exp\left(-\frac{kt}{1+R}\right) \tag{58}$$

for oscillator 3. We have expressed the relations in terms of

$$A(t) = c(t)\exp(-kt) \tag{59}$$

because it is the variable used in the literature.

From equations (44)–(48) and (49)–(52) and the last column of Table 1, we have plotted Figures 1–3. These figures show R versus the reduced energy $E_R = E/V_m$ (V_m is defined as $-A^2/4B$ for the oscillator $(x')^2 + Ax^2 + Bx^4 = E$). One sees that the suitable choice of R is dependent on the initial conditions. Nevertheless for each oscillator, over a wide range of energies, the value of R is nearly constant. These are, roughly, $R = \frac{1}{2}$ for oscillator 1 (high energy), $R = \frac{5}{8}$ for oscillator 2 (low energy), and $R = 4$ when $E_1 < 0$ and $R = R_{cn} = \frac{1}{2}$ when $E_1 > 0$ (R_{cn} is the value of R if one assumes a solution of type cn with $\frac{1}{2} < \mu < 1$)

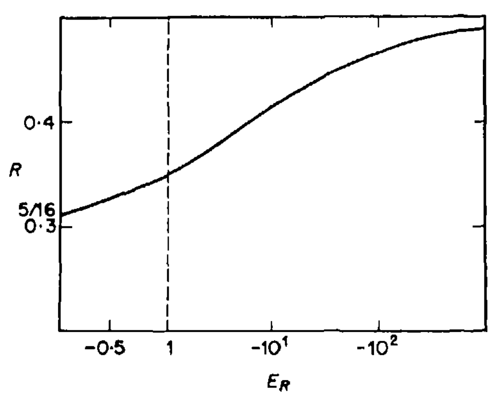

Figure 1. R vs. E_R for oscillator 1 (solution cn).

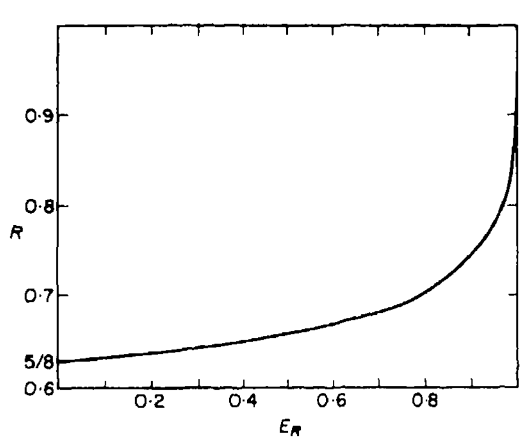

Figure 2. R vs. E_R for oscillator 2 (solution cd).

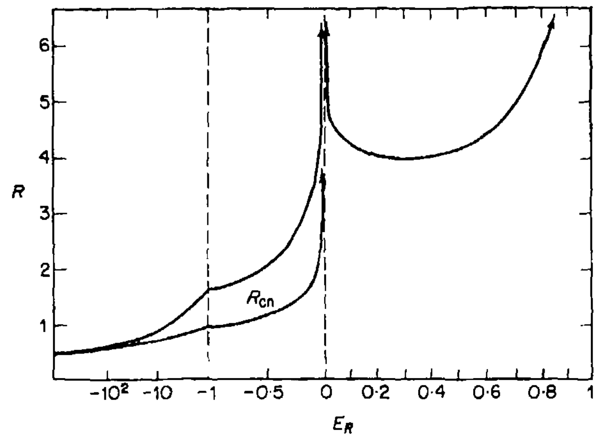

Figure 3. R vs. E_R for oscillator 3 (solution dn). The line labelled R_{cn} has been calculated with the cn solution.

for oscillator 3. Tables 2–5 show the results obtained with these values of R. However, as noted above, for certain initial conditions these values are not optimum. There is a good example in the second case of Table 4 ($E>0$): the choice $R_{cn}=\frac{1}{2}$ is not good, far better is $R_{cn}=\frac{3}{4}$. This could be foreseen from Figure 3 because for $t\cong 0$, $E_{1R}\cong 8$ and then $R_{cn}\cong\frac{3}{4}$.

The relation (56) with $R=\frac{1}{2}$ is given in references [1, 3], in a slightly different form. The relations (57) and (58) are, to our knowledge, new in the literature on the subject.

Tables 3 and 4 show the results that one gets by using the Mendelson method [7]. This method is not applicable to oscillator 3. It gives a formula for the amplitude decay [8] that can also be derived from equations (56) and (57). One can write these formulas as

$$a(t)=a(0)\exp(-kt)\left\{\frac{1+\beta\varepsilon a^2(0)}{1+\beta\varepsilon a^2(t)}\right\}^{R/2}, \tag{60}$$

TABLE 2

Oscillator 1; *values of* $|x_{max}(t)|/x(0)$; *"linear" means that the amplitude decay has the form of* $\exp(-kt)$; *the non-linearity of each case is given by* $c_3x^2(0)/c_1$

Case	Time	Runge-Kutta	Sophisticated mode	Simple mode $R=\frac{1}{2}$	Mendelson	Linear
	2·43	0·90	0·90	0·91	0·93	0·89
$b=0{\cdot}1$	4·97	0·81	0·81	0·82	0·85	0·78
$c_1=1$	7·59	0·72	0·72	0·73	0·77	0·69
$c_3=1$	10·30	0·64	0·64	0·65	0·69	0·60
$x(0)=1$	13·10	0·56	0·56	0·58	0·61	0·52
	13·96	0·49	0·49	0·51	0·54	0·45
$b=0{\cdot}5$						
$c_1=1$	2·62	0·56	0·57	0·59	0·64	0·54
$c_3=1$	5·63	0·27	0·28	0·29	0·32	0·25
$x(0)=1$						
$b=0{\cdot}5$	0·77	0·88	0·88	0·88		0·85
$c_1=1$	1·64	0·76	0·76	0·76	Inapplicable,	0·69
$c_3=1$	2·65	0·63	0·64	0·64	too	0·53
$x(0)=5$	3·85	0·51	0·52	0·53	non-linear	0·40
	5·29	0·39	0·40	0·41		0·28

TABLE 3

Oscillator 2; *values of* $|x_{max}(t)|/x(0)$; *linear means the same as in Table* 2; *the non-linearity of each case is given by* $c_3x^2(0)/c_1$

Case	Time	Runge–Kutta	Sophisticated mode	Simple mode $R=\frac{5}{8}$	Mendelson	Linear
$b=0{\cdot}1$	5·3	0·70	0·68	0·69	0·72	0·77
$c_1=1$	9·0	0·56	0·55	0·56	0·58	0·64
$c_3=-1$	12·5	0·46	0·45	0·46	0·48	0·54
$x(0)=0{\cdot}96$	15·9	0·39	0·38	0·38	0·40	0·45
	19·2	0·32	0·32	0·32	0·33	0·38
$b=0{\cdot}1$						
$c_1=1$	4·1	0·35	0·34	0·35	0·34	0·37
$c_3=-1$						
$x(0)=0{\cdot}80$	7·4	0·15	0·15	0·15	0·15	0·16
$b=0{\cdot}1$	4·3	0·75	0·75	0·76	0·77	0·81
$c_1=1$	7·9	0·61	0·61	0·62	0·63	0·67
$c_3=-1$	11·4	0·51	0·51	0·51	0·52	0·57
$x(0)=0{\cdot}84$	14·7	0·43	0·43	0·43	0·44	0·48
	18·0	0·36	0·36	0·36	0·37	0·41

TABLE 4

Oscillator 3; $E(0)>0$; *values of* $|x_{max}(t)/x(0)|$; "*linear*" *means the same as in Table* 2; *the non-linearity of each case is given by* $c_3x^2(0)/c_1$

Case	Time	Runge–Kutta	Sophisiticated mode	Simple mode $R=\frac{1}{2}$	Simple mode $R=\frac{3}{4}$	Linear	$E(t)$
$b=0{\cdot}5$	0·82	0·88	0·88	0·88		0·82	
$c_1=1$	1·78	0·75	0·75	0·75		0·64	
	2·93	0·63	0·63	0·62		0·48	$E(t)>0$
$c_3=1$	4·41	0·50	0·50	0·49		0·33	
$E(0)>0$	6·49	0·37	0·37	0·36		0·20	
	10·83	0·26	0·26	0·22		0·07	$E(t)>0$
$x(0)=5$	13·02	0·16	0·16				
$b=0{\cdot}1$	2·47	0·94	0·94	0·93	0·94	0·88	
$c_1=-1$	5·27	0·87	0·87	0·86	0·88	0·77	$E(t)>0$
$c_3=1$	8·51	0·82	0·81	0·78	0·82	0·65	
$E(0)>0$	12·47	0·75	0·75	0·71	0·75	0·54	
$x(0)=2$	18·06	0·70	0·68	0·62	0·67	0·41	$E(t)<0$

where

$$a(t)=x(0)A(t), \qquad \varepsilon=c_3/(c_1-k^2), \tag{61}$$

and where $\beta=1$ for oscillator 1 and $\beta=\frac{1}{2}$ for oscillator 2. For εa^2 small (weakly non-linear motion), one can make the approximation

$$(1+\beta\varepsilon a^2)^{-R}=1-R\beta\varepsilon a^2 \tag{62}$$

and equation (60) then becomes

$$a(t)=a(0)\exp(-kt)\{1+R\beta\varepsilon a^2(0)[\exp(-2kt)-1]\}^{1/2}. \tag{63}$$

TABLE 5

Oscillator 3; $E(0)<0$; values of $x_{max}(t)$ and $x_{low}(t)$; for "linear" the decay is evaluated for a simple harmonic well centered at the position of the minimum

Case	Time	Runge-Kutta	Sophisticated mode	Simple mode $R=4$	Linear
$b=0{\cdot}1$	2·73	0·39	0·39	0·38	0·43
	5·58	1·32	1·32	1·33	1·30
$c_1=-1$	8·01	0·57	0·59	0·58	0·63
$c_3=1$	10·53	1·26	1·26	1·26	1·24
	12·86	0·68	0·69	0·70	0·73
$x(0)=1{\cdot}40$	15·26	1·21	1·21	1·20	1·19
	17·54	0·76	0·77	0·79	0·80
$E(0)<0$	19·87	1·17	1·17	1·15	1·15
$b=0{\cdot}5$	2·24	0·67	0·68	0·68	0·78
$c_1=-1$	4·80	1·14	1·18	1·14	1·12
$c_3=1$	6·94	0·91	0·94	1·04	0·99
$x(0)=1{\cdot}40$	9·31	1·05	1·08	0·91	1·04
$E(0)<0$	11·55	0·97	0·98	1·21	1·04
	13·82	1·02	1·06	0·72	1·01

With $\varepsilon=c_3/c_1$, and choosing R in such a way that $R\beta=\frac{3}{8}$, one gets the Mendelson formula. From Figures 1 and 2, our selection of R for weakly non-linear motion should be $R\beta=\frac{5}{16}$. This value is slightly better than $R\beta=\frac{3}{8}$ for oscillator 1, but it is worse for oscillator 2. But in the linear limit and for not very large damping, i.e., in the region where these formulae work well, the differences are very small.

4.2. SOPHISTICATED MODE RESULTS

For the sophisticated mode results, we make no assumption about $R(\mu^2)$ and follow the procedure of Christopher [2]. First, we express ω'/ω in terms of μ and μ' as we did for oscillator 2, equation (31). For the other oscillators, equation (31) takes the form

$$\omega'/\omega=\mu\mu'/(\tfrac{1}{2}-\mu^2) \tag{64}$$

for oscillator 1, and

$$\omega'/\omega=\mu\mu'/(2-\mu^2) \tag{65}$$

for oscillator 3.

To express $R(\mu^2)$ in series of powers of μ^2 one can use equations (45)-(47) and (49)-(51). The $\mathrm{E}(\mu^2)$ and $\mathrm{K}(\mu^2)$ series are given in reference [6]. One finds, after some algebra,

$$R(\mu^2)=(5/16)+(29/128)\mu^2+(359/2048)\mu^4+(289/2048)\mu^6+(15311/131072)\mu^8 \tag{66}$$

for oscillator 1,

$$R(\mu^2)=(5/8)+(11/64)\mu^2+(71/1024)\mu^4+(9/256)\mu^6+(313/16384)\mu^8 \tag{67}$$

for oscillator 2, and

$$R(\mu^2)=(4/\mu^2)-1+(3/32)\mu^2+(3/32)\mu^4+(349/4096)\mu^6+(314/4096)\mu^8 \tag{68}$$

for oscillator 3 ($E_1 < 0$). Substituting equations (64) and (66) into equation (55) and integrating gives

$$c(t) = c(0)\left\{\frac{1-2\mu^2(t)}{1-2\mu^2(0)}\right\}^{0\cdot 2488} \exp\{G[\mu^2(t)] - G[\mu^2(0)]\} \tag{69}$$

with

$$G(\mu^2) = 0\cdot 1851\mu^2 + 0\cdot 719\mu^4 + 0\cdot 0374\mu^6 + 0\cdot 0208\mu^8 \tag{70}$$

for oscillator 1 (and oscillator 3 with $E_1 > 0$, too), similarly substituting equations (31) and (67) gives

$$c(t) = c(0)\left\{\frac{1+\mu^2(t)}{1+\mu^2(0)}\right\}^{0\cdot 2472} \exp\{G[\mu^2(t)] - G[\mu^2(0)]\} \tag{71}$$

with

$$G(\mu^2) = 0\cdot 0653\mu^2 + 0\cdot 0103\mu^4 + 0\cdot 0047\mu^6 + 0\cdot 0014\mu^8 \tag{72}$$

for oscillator 2, and substituting equations (65) and (68) gives

$$c(t) = c(0)\,\frac{\mu(0)}{\mu(t)}\left\{\frac{2-\mu^2(t)}{2-\mu^2(0)}\right\}^{1777/1024} \exp\{G[\mu^2(t)] - G[\mu^2(0)]\} \tag{73}$$

with

$$G(\mu^2) = (1265/2048)\mu^2 + (1169/8192)\mu^4 + (977/25776)\mu^6 + (314/32768)\mu^8 \tag{74}$$

for oscillator 3 ($E_1 < 0$).

With these equations and constraint 4 one can obtain the value of c for each oscillator. The results that one gets with this method are very good, as can be seen in Tables 2-5; as said before, they are better for large frequencies and small damping.

5. CONCLUSIONS

The work presented here is intended to provide approximate solutions for the simplest possible non-linear damped oscillators: i.e., solutions of equation (1). The equation of motion has to date not been solved exactly in a simple analytical way. We have used a method of approximation due to Christopher [2] for the case $c_1 > 0$, $c_3 > 0$ and extended it to the cases $c_1 > 0$, $c_3 < 0$ (bounded motion) and $c_1 < 0$, $c_3 > 0$. We have found that there exist expressions, some quite simple, for the amplitude decay which prove to be accurate over a wide range of the parameters of the problem.

REFERENCES

1. T. D. BURTON 1983 *Journal of Sound and Vibration* **87**, 535-541. On the amplitude decay of strongly non-linear damped oscillators.
2. P. A. T. CHRISTOPHER 1973 *International Journal of Control* **17**, 597-608. An approximate solution to a strongly non-linear, second order, differential equation.
3. P. A. T. CHRISTOPHER and A. BROCKLEHURST *International Journal of Control* **19**, 831-839. A generalized form of an approximate solution to a strongly non-linear, second order, differential equation.
4. N. MINORSKY 1962 *Nonlinear Oscillations*. Malabar, Florida: Robert E. Krieger, first edition, 1983 reprint.
5. S. BRAVO YUSTE 1984 *Tesis de Licenciatura, University of Extremadura*. Aplicación de las funciones elípticas a los osciladores cuárticos amortiguados.

6. M. ABRAMOWITZ and I. A. STEGUN (editors) 1972 *Handbook of Mathematical Functions.* New York: Dover.
7. K. S. MENDELSON 1970 *Journal of Mathematical Physics* **11**, 3413–3415. Perturbation theory for damped nonlinear oscillators.
8. R. E. MICKENS 1981 *An Introduction to Nonlinear Oscillations.* Cambridge: Cambridge University Press.