Average shape of fluctuations for subdiffusive walks

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We study the average shape of fluctuations for subdiffusive processes, i.e., processes with uncorrelated increments but where the waiting time distribution has a broad power-law tail. This shape is obtained analytically by means of a fractional diffusion approach. We find that, in contrast with processes where the waiting time between increments has finite variance, the fluctuation shape is no longer a semicircle: it tends to adopt a tablelike form as the subdiffusive character of the process increases. The theoretical predictions are compared with numerical simulation results.

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I. INTRODUCTION

Complex systems are often described by their lack of a characteristic length or time scale over many orders of magnitude, which gives rise to events whose distribution in sizes is a power law with no characteristic size (fractal behavior). Examples are everywhere [1]: seismic activity, turbulence, solar flares, Brownian motion, length of rivers and blood vessels, ... In particular, a power-law distribution with no characteristic *temporal* size events appears in the analysis of stock price changes [2,3], river floods [4] , Barkhausen noise [5,6], glassy systems [7–9], atomic cooling [10], and fluorescence of quantum dots [11,12]. In these cases the resulting dynamics is strongly intermittent, with bursts of activity separated by long quiescent intervals.

When these temporal intervals are waiting times δt between jumps of size δx, then the stochastic process $x(t)$ can be seen as the trajectory of a subdiffusive random walker (provided that the variance of δx is finite). A typical subdiffusive trajectory is shown in Fig. 1. Systems that exhibit anomalous subdiffusion characterized by an anomalous Fick's second law

$$\langle x^2(t)\rangle \sim \frac{2K_\gamma}{\Gamma(1+\gamma)} t^\gamma, \tag{1}$$

where $0<\gamma<1$ are ubiquitous in nature [13,14]. [K_γ is the (generalized) diffusion constant and γ is the anomalous diffusion exponent.] Also, models based on subdiffusive random walkers are useful for understanding complex systems. Two nice examples are the "trap model," proposed to explain aging in disordered systems [7–9], and the comb model, to understand diffusion phenomena in complex structures such as percolation clusters [15].

Recently [6,5,16,18] the study of the average shape of the fluctuations of stochastic processes $x(t)$ has been considered as a useful tool to gain insight into the system that generates the fluctuation. Thus, it has been argued [18] that the average shape of fluctuations is a better tool for discriminating between theories than critical exponents.

In Ref. [16] Baldassarri *et al.* consider the average shape for a stochastic process of the form $x(t+1)=x(t)+\delta x$, where δx is a random variable. Let us denote the average shape of fluctuations of time span T by $\langle x(t)\rangle_T$. Baldassarri *et al.* find that the shape follows the scaling law

$$\langle x(t)\rangle_T = T^{\alpha/2} f(t/T), \tag{2}$$

where α is the diffusion exponent and f is a *semicircle* whenever δx follows a distribution with finite variance (Gaussian walks) or a distribution $\lambda(\delta x)$ with a broad power-law tail, $\lambda(\delta x)\sim(\delta x)^{-\mu-1}$ with $0<\mu<2$, so that the variance is infinite (Lévy flights). This result had already been obtained by Fisher for Gaussian walks [19][Sec. 7.1]. The fact that the average shape of fluctuations is a semicircle for both Gaussian walks and Lévy flights is a nicely surprising result that led us to wonder to what extent it might hold for other walks with uncorrelated jumps. In particular, we investigated the average shape of subdiffusive stochastic processes (subdiffusive walks) of the form

$$x(t+\delta t)=x(t)+\delta x, \tag{3}$$

where δx are uncorrelated increments that follow a distribution with finite variance and δt is a random variable whose distribution $\psi(\delta t)$ has a broad power-law tail: $\psi(\delta t) \sim(\delta t)^{-1-\gamma}$ with $0<\gamma<1$. Figure 1 shows the position $x(t)$ for a stochastic process of this kind with $\gamma=0.9$ and with jumps of unit length ($\delta=\pm 1$) made with equal probabilities.

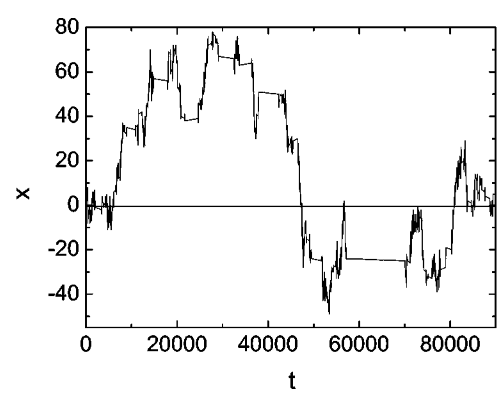

FIG. 1. Subdiffusive trajectory with $\delta x=\pm 1$ and $\gamma=0.9$.

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II. AVERAGE FORM OF A SUBDIFFUSIVE FLUCTUATION

The average shape of a fluctuation can be calculated through the expression [16,17]

$$\langle x(t)\rangle_T = \lim_{x_0\to 0^+} \frac{\overline{\int_0^\infty dx\, x\, \Omega(x,t|\{x(0)\};x_0,0,x_0,T)}}{\int_0^\infty dx\, \Omega(x,t|\{x(0)\};x_0,0,x_0,T)} \qquad (4)$$

where $\Omega(x,t|\{x(0)\};x_0,0,x_0,T)$ is the probability that the walker with trajectory $\{x(0)\}$ for $t<0$ is at x at time t provided that he was at $x_0>0$ at time 0 and at x_0 at time T without ever touching the axis at $x=0$ in the time interval $(0,T)$. The upper line in Eq. (4) means average over all the trajectories $\{x(0)\}$ that reach x_0 at time $t=0$. Let $F(x,t|\{x(t')\};x',t')$ be the probability that the walker with trajectory $\{x(t')\}$ for $t<t'$ and who was at $x'>0$ at time t' reaches $x>0$ at time t without ever touching the axis at $x=0$; let $\Omega(x,t|x_0,0,x_0,T)$ be the probability that the walker is at x at time t provided that he was at $x_0>0$ at time 0 and at x_0 at time T without ever touching the axis at $x=0$ in the time interval $(0,T)$; and let $F(x,t|x',t')$ be the probability that the walker who was at x' at time t' reaches $x>0$ at time t without ever touching the axis at $x=0$. For walks without memory (Markovian walks) $\Omega(x,t|\{x(0)\};x_0,0,x_0,T)=\Omega(x,t|x_0,0,x_0,T)$, $F(x,t|\{x(t')\};x',t')=F(x,t|x',t')$, and one can write $\Omega(x,t|x_0,0,x_0,T)$ as $F(x,t|x_0,0)F(x,T-t|x_0,0)$, so that Eq. (4) becomes [16,17]

$$\langle x(t)\rangle_T = \lim_{x_0\to 0^+} \frac{\int_0^\infty dx\, x\, F(x,t|x_0,0)F(x,T-t|x_0,0)}{\int_0^\infty dx\, F(x,t|x_0,0)F(x,T-t|x_0,0)}. \qquad (5)$$

This equation *is not exact* for subdiffusve walks because they are not Markovian. However, for subdiffusive walks, the memory, i.e., the effect of the fact that at t' the particle was at x' on the probability that the particle at time $t>t'$ is at x, decays as $(t-t')^{-\gamma}/\Gamma(1-\gamma)$ [13]. This implies that the approximation of $F(x,t|\{x(t')\};x',t')$ by $F(x,t|x',t')$, and, consequently, the accuracy of Eq. (5) for subdiffusive walks, improves when $t-t'$ increases and γ is close to 1.

The probability $F(x,t|x_0,0)$ can be calculated by means of the method of images $F(x,t|x_0,0)=P(x-x_0,t)-P(-x-x_0,t)$ [20,21], $P(x-x_0,t)$ being the probability density that the free process (without boundary conditions) that at time $t\leqslant 0$ was at x_0 is at x at time t. For subdiffusive processes, and for $t^{-\gamma}/\Gamma(1-\gamma)\ll 1$ [22], $P(x,t)$ can be written in terms of Fox's H function as [13]

$$P(x,t)=\frac{1}{\sqrt{4K_\gamma t^\gamma}} H_{11}^{10}\left[\frac{|x|}{\sqrt{K_\gamma t^\gamma}}\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right]. \qquad (6)$$

Taking into account that the Laplace transform of $P(x,t)$ is

$$P(x,u)=\frac{u^{\gamma/2-1}}{\sqrt{4K_\gamma}}\exp(-\sqrt{u^\gamma/K_\gamma}|x|) \qquad (7)$$

one finds for $x\geqslant 0$

$$F(x,u|x_0,0)=\frac{u^{\gamma/2-1}}{\sqrt{4K_\gamma}}\{e^{-a(x_0-x)}[\Theta(x)-\Theta(x-x_0)] + e^{-a(x-x_0)}\Theta(x-x_0)-e^{-a(x+x_0)}\}, \qquad (8)$$

where $a\equiv\sqrt{u^\gamma/K_\gamma}$ and $\Theta(x)$ is the Heaviside step function. As we are interested in the limit $x_0\to 0$ with $x>x_0$, we expand the term inside the bracket in powers of x_0 and get

$$F(x,u|x_0\to 0,0)=x_0\frac{u^{\gamma-1}}{K_\gamma}e^{-\sqrt{u^\gamma/K_\gamma}x}, \qquad (9)$$

which implies

$$F(x,t|x_0\to 0,0)=\frac{x_0}{K_\gamma t^\gamma}H_{11}^{10}\left[\frac{x}{\sqrt{K_\gamma t^\gamma}}\middle|\begin{matrix}(1-\gamma,\gamma/2)\\(0,1)\end{matrix}\right]. \qquad (10)$$

Inserting this expression into Eq. (5) and carrying out the integrations [23] one finds that the average shape of a subdiffusive stochastic process (subdiffusive random walk) is given by

$$\langle x(t)\rangle_T = T^{\gamma/2}f_\gamma(t/T)=\sqrt{K_\gamma T^\gamma}g_\gamma(t/T), \qquad (11)$$

where

$$g_\gamma(t/T)=\frac{\left(\frac{t}{T}\right)^{\gamma/2}H_{22}^{11}\left[\left(\frac{t/T}{1-t/T}\right)^{\gamma/2}\middle|\begin{matrix}(-1,1),\left(1-\gamma,\frac{\gamma}{2}\right)\\(0,1),\left(0,\frac{\gamma}{2}\right)\end{matrix}\right]}{H_{22}^{11}\left[\left(\frac{t/T}{1-t/T}\right)^{\gamma/2}\middle|\begin{matrix}(0,1),\left(1-\gamma,\frac{\gamma}{2}\right)\\(0,1),\left(\frac{\gamma}{2},\frac{\gamma}{2}\right)\end{matrix}\right]}. \qquad (12)$$

In Fig. 2 we plot the (normalized) average shape of fluctuations for several classes of subdiffusive processes. We see that the shape tends to a tablelike form as γ decreases, i.e., as the subdiffusive character of the process increases.

Of course, for $\gamma\to 1$ one recovers the Gaussian result $f_1(t/T)=\sqrt{16D/\pi}\sqrt{t/T(1-t/T)}$ [16,19] because the upper and lower Fox's H functions in Eq. (12) become $2z/\pi(1+z^2)^2$ and $z/2\sqrt{\pi}(1+z^2)^{3/2}$, respectively.

The area $N(T)$ of the average fluctuation of duration T is given by

$$N(T)=\int_0^1 ds\langle x(sT)\rangle_T, \qquad (13)$$

where $s=t/T$. From Eq. (11) one finds

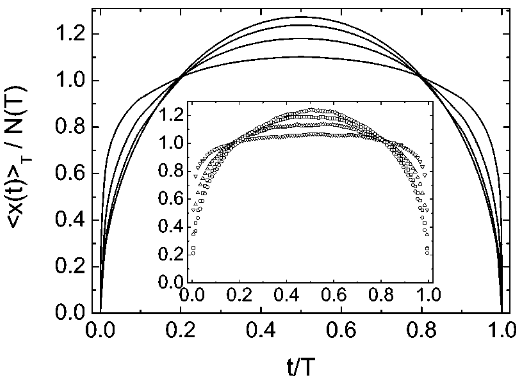

FIG. 2. Normalized average fluctuation for subdiffusive processes for several values of γ. The shape is normalized so that its area is 1, i.e., we plot $\langle x(t)\rangle_T/N(T)$. The lines are the theoretical result for (at $t/T=1/2$ and from top to bottom) $\gamma=1,0.9,0.75,0.5$. Inset: simulation results for these same values of γ.

$$N(T)=n_\gamma\sqrt{K_\gamma T^\gamma}, \tag{14}$$

where $n_\gamma=\int_0^1 ds g_\gamma(s)$. Then, the normalized average fluctuation is given by

$$\frac{\langle x(t)\rangle_T}{N(T)}=\frac{g_\gamma(t/T)}{n_\gamma}. \tag{15}$$

We have not been able to calculate n_γ analytically. In Table I we give some values evaluated numerically.

III. SIMULATION

We carried out simulations of the fluctuations of the stochastic process (3) where δx takes the values $+1$ or -1 with equal probabilities and where the waiting time δt between jumps follows the Pareto distribution $\psi(t)=\gamma/(1+t)^{1+\gamma}$. In this case, the diffusion constant K_γ is given by Ref. [13] $K_\gamma=1/[2\Gamma(1-\gamma)]$. The simulation results follow the pattern found in the precedent section: the fluctuation shape tends toward a tablelike form as γ decreases (see inset in Fig. 2).

In Figs. 3 and 4 we compare the theoretical predictions and the simulation results for $\gamma=0.9$ and $\gamma=0.75$, respectively. The agreement is reasonable. We attribute the differences to, first, the approximate nature of Eq. (5) for non-Markovian processes and, second, to a slow convergence that requires longer T to set in. This can be clearly appreciated in Fig. 5 where one sees that the well-established theoretical semicircular shape is approached, although very gradually, as T increases . It is even more gradual for smaller γ. We have not explored larger values for T because of the excessive computer time required.

In the simulations we also calculated the area $N(T)$ of the

TABLE I. The coefficient $n_\gamma=\int_0^1 ds g_\gamma(s)$ calculated by numerical integration.

γ	1/4	1/3	1/2	2/3	3/4	4/5	9/10	1
n_γ	0.493	0.501	0.537	0.612	0.668	0.708	0.798	$\sqrt{\pi}/2$

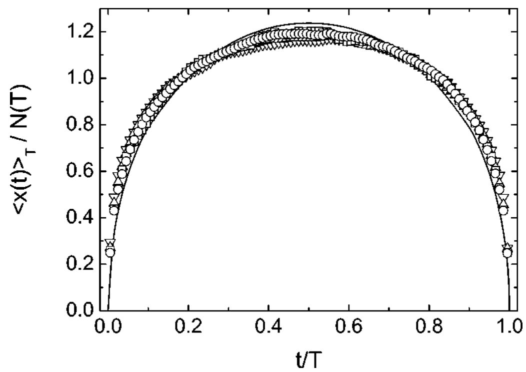

FIG. 3. Normalized average fluctuation for the subdiffusive process with $\gamma=0.9$. The line is the theoretical result and the symbols are simulation results for $T=10^4,10^5,10^6,10^7,10^8$.

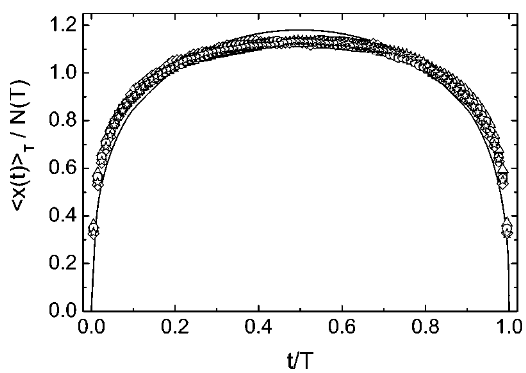

FIG. 4. Normalized average fluctuation for the subdiffusive process with $\gamma=0.75$. The line is the theoretical result and the symbols are simulation results for $T=10^4,10^5,10^6,10^7,10^8$.

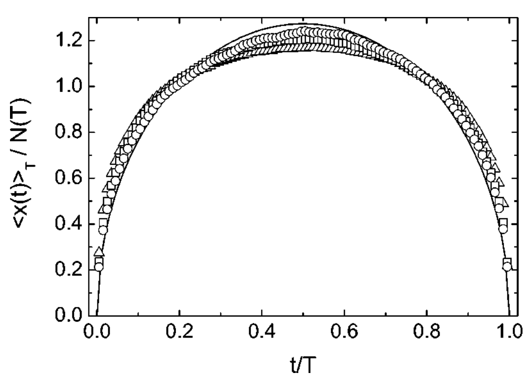

FIG. 5. Normalized average fluctuation for a process with $\gamma=1$. The line is the semicircular form theoretical result and the symbols are simulation results for $T=10^3$ (triangles), 10^5 (squares), and 10^7 (circles).

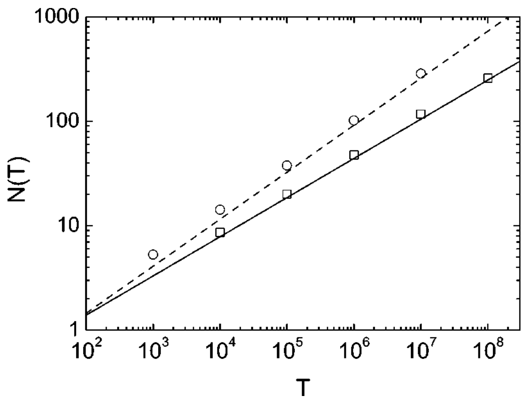

FIG. 6. Area $N(T)$ of the fluctuation for $\gamma=0.75$ (circles) and $\gamma=0.9$ (squares). The lines are the corresponding theoretical prediction; see Eq. (14).

fluctuation. In Fig. 6 we plot $N(T)$ for several values of T and for two values of γ. The agreement between simulation and theory is reasonable again.

IV. CONCLUSIONS

We have analyzed the average shape of the fluctuations of time series generated by the stochastic process (3) with a power-law waiting time distribution $\psi(t)\sim t^{-1-\gamma}$ where $0<\gamma<1$ (subdiffusive walks). Although the spatial increments δx in this stochastic process are uncorrelated, we find that, in contrast with Gaussian walks and Lévy flights, the average shape of a fluctuation is no longer a semicircle: its form becomes flatter at the top and steeper at the extremes as the subdiffusive character of the process increases (i.e., as γ decreases).

Some possible sequels of the present work are obvious: one could consider the effect of correlated increments δx (as done by Baldassarri *et al.* [16]) or investigate the average shape of a fluctuation for Lévy walks (in which $|\delta x|$ and δt are proportional and follow a broad power-law tail). However, for these cases, the task of getting analytical results, even approximate, such as those reported in the present paper certainly looks formidable.

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