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# Multiparticle trapping problem in the half-line

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## Abstract

A variation of Rosenstock's trapping model in which  $N$  independent random walkers are all initially placed upon a site of a one-dimensional lattice in the presence of a *one-sided* random distribution (with probability  $c$ ) of absorbing traps is investigated. The probability (survival probability)  $\Phi_N(t)$  that no random walker is trapped by time  $t$  for  $N \gg 1$  is calculated by using the extended Rosenstock approximation. This requires the evaluation of the moments of the number  $S_N(t)$  of distinct sites visited in a *given* direction up to time  $t$  by  $N$  independent random walkers. The Rosenstock approximation improves when  $N$  increases, working well in the range  $Dt \ln^2(1-c) \ll \ln N$ ,  $D$  being the diffusion constant. The moments of the time (lifetime) before any trapping event occurs are calculated asymptotically, too. The agreement with numerical results is excellent. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Survival of Brownian particles in a medium populated with randomly distributed static traps is a fundamental problem (the “trapping” problem) of random walk theory that has been an active area of research for decades with many applications in physics and chemistry [1–4]. The origin of this problem can be traced back to Smoluchowski's theory of coagulation of colloidal particles [1–3]. It has now become a basic model of widespread interest in areas such as trapping of mobile defects in crystals with point

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sinks [5–7], the kinetics of luminescent organic materials [7], the kinetics of photosynthetic light energy to oxygen conversion [8,9], anchoring of polymers by chemically active sites [10,11], atomic diffusion in glasslike materials [12,13] and many more [14].

This paper is devoted to a variation of the so-called Rosenstock trapping problem on a one-dimensional substrate. Usually, the one-dimensional Rosenstock trapping problem (which we will call the “two-sided” Rosenstock trapping problem for reasons that will be apparent later on) is stated as follows [1–3]. A one-dimensional lattice is filled with a random distribution of static traps; then, one ( $N = 1$ ) random walker is placed initially ( $t = 0$ ) at a given site of the lattice; it starts to diffuse and, eventually, is caught by a trap. In this paper we study a different but closely related trapping problem (which we will call the “one-sided” Rosenstock trapping problem) in which (i) only a *half-line* of a one-dimensional lattice is filled with a random distribution of static traps with concentration  $c$  (this process could mimic the excitation or production of defects in one side of a fiber by irradiation, the other side being shielded) and (ii)  $N$  independent random walkers are placed initially ( $t = 0$ ) at the contact point ( $x = 0$ ) between the two half-lines that is taken as origin. These random walkers start to diffuse, and eventually one of them is trapped at the nearest site occupied by a trap (or deactivates it). The statistical quantities of main interest are the survival probability,  $\Phi_N(t)$ , defined as the probability that no random walker has been trapped by time  $t$ , and the lifetime,  $T_N$ , defined as the average time at which the first random walker of the set of  $N$  arrives at a trap site. To our knowledge, this is the first multiparticle ( $N \neq 1$ ) Rosenstock trapping problem ever studied. A good reason for this is that the multiparticle versions of the trapping problem are much more difficult to solve than the trapping problems with a single particle.

This fact will be evident in this paper: the present trapping problem is elementary for  $N = 1$  and, in this case, we will report the main results for the sake of completeness only (see Section 3). On the other hand, the multiparticle version is much more involved (see Sections 4 and 5). As an exact evaluation of  $\Phi_N(t)$  for  $N > 1$  is elusive, we have resorted to asymptotic analysis techniques. In particular, for  $N \gg 1$ , we have used the extended Rosenstock approximation (or truncated cumulant approximation) [1,2,14–16]. This requires to find the first moments  $\langle S_N^m(t) \rangle$ ,  $m = 1, 2, \dots$ , of  $S_N(t)$ , the number of sites situated to the “right” of the origin  $x = 0$  that were visited up to time  $t$  by  $N$  random walkers which started at the site  $x = 0$  at time  $t = 0$ . Note that  $S_N(t)$  is just the maximum distance reached by any of the  $N$  random walkers in the  $+x$  direction from the origin by time  $t$ , i.e.,  $S_N(t)$  is the one-sided span of the  $N$ -particle random walk. The problem of evaluating the first moment  $\langle S_N(t) \rangle$  has already been addressed in Refs. [17–19] explicitly and in Refs. [20–26] implicitly. The quantity studied in these last references was the number of distinct sites explored by  $N$  random walkers on the one-dimensional lattice in *either* direction,  $\tilde{S}_N(t)$ , but  $\langle S_N(t) \rangle = \langle \tilde{S}_N(t) \rangle / 2$ . However, little is known about higher moments of  $S_N(t)$ ,  $\langle S_N^m(t) \rangle$ , except for some rough estimates [18,19,23]. The idea of evaluating the survival probability for the multiparticle trapping problem by using the moments of  $\tilde{S}_N(t)$  into the Rosenstock approximation was suggested by Larralde et al. in Ref. [23], although, to the best of our knowledge, it

has not been implemented perhaps for the lack of precise expressions for the moments  $\langle \tilde{S}_N^m(t) \rangle$ . Therefore, in order to use the Rosenstock approximation in our one-sided multiparticle trapping problem, we must find rigorous asymptotic results for the moments of  $S_N(t)$ . This is another objective (important in itself) of this present work that, besides, we hope can illuminate how to deal with the evaluation of the moments of  $\tilde{S}_N(t)$  for other substrates.

The paper is organized as follows. In Section 2, we recall the connection between the survival probability  $\Phi_N(t)$  and the so-called fixed trap survival probability (i.e., the survival probability when the trap is placed at a given distance) and give the basics of the extended Rosenstock approximation. The one-sided trapping problem for  $N = 1$  is addressed in Section 3. In Section 4, we calculate the moments of  $S_N(t)$  in the form of an asymptotic series in which the corrective terms decay as powers of  $1/\ln N$ . The evaluation of  $\langle S_N^m(t) \rangle$  is a necessary prerequisite for the implementation, in Section 5, of the Rosenstock approximation for  $\Phi_N(t)$ . In this section we compare  $\Phi_N(t)$  as given by the extended Rosenstock approximation with numerical results. The moments of the lifetime  $T_N$  and its variance are evaluated in Section 6. The paper ends with some conclusions and remarks.

## 2. One-sided trapping model and Rosenstock approximation

The one-sided Rosenstock trapping model is defined as follows: (i) quenched traps are randomly distributed on the right-hand side of a one-dimensional lattice ( $x > 0$ ) with concentration  $c$  ( $1 - c \equiv p$ ); (ii) the random walkers are placed initially upon site  $x = 0$  which divides the randomly filled trapping half-line and the empty one; and (iii) the traps are irreversible, that is, a walker encountering a trap is killed there. Then, the survival probability is given by  $\Phi_N(t) = \sum_{r=1}^t p^r P_N(t|r)$ , where  $P_N(t|r)$  is the probability that the span of the  $N$  random walkers in the positive direction (the largest distance reached by any of the  $N$  random walkers for  $x > 0$ ) is equal to  $r$  after  $t$  time steps [1]. Let  $\Gamma_N(t|r)$  be the probability that the site  $x = r$  has not been visited by any of the  $N$  random walkers by time  $t$  (the so-called fixed-trap survival probability). Then, in the continuous limit,

$$\Phi_N(t) = \int_0^\infty p^r \frac{d\Gamma_N(t|r)}{dr} dr, \quad (1)$$

where we have used the relationship  $P_N(t|r) = d\Gamma_N(t|r)/dr$  between the one-sided span distribution  $P_N(t|r)$  and the fixed-trap survival probability  $\Gamma_N(t|r)$ . For  $N$  independent random walkers one has  $\Gamma_N(t|r) = [\Gamma(t|r)]^N$  where  $\Gamma(t|r) = \Gamma_1(t|r)$  is the probability that distance  $r$  has not been reached by a single random walker by time  $t$ . For the one-sided diffusion process it is well known that [1,2]

$$\Gamma(t|r) = \operatorname{erf}\left(\frac{r}{\sqrt{4Dt}}\right). \quad (2)$$

The extended Rosenstock approximation (or truncated cumulant expansion) is now a standard approach [1–3] to the Rosenstock trapping problem, which was first proposed by Zumofen and Blumen [15] and that we recall here for the sake of reference. From the definition of  $S_N(t)$  as the number of distinct sites on the positive half-line visited up to time  $t$  by  $N$  independent random walkers that started at  $x=0$  at time  $t=0$  (note that this means  $S_N(0)=0$ ), the survival probability of the  $N$  random walkers is  $\Phi_N(t) = \langle p^{S_N(t)} \rangle \equiv \langle \exp[S_N(t) \ln p] \rangle$ . The average in this equation is performed over all realizations of the random walkers' exploration of the lattice up to time step  $t$ . The well-known cumulant expansion technique [1,2] allows an alternative form of  $\Phi_N(t)$  as an infinite series expansion

$$\Phi_N(t) = \exp \left[ \sum_{n=1}^{\infty} \frac{\kappa_n (\ln p)^n}{n!} \right], \tag{3}$$

where  $\kappa_n, n=1,2,\dots$  denote the cumulants of  $S_N(t)$ :  $\kappa_1 = \langle S_N(t) \rangle, \kappa_2 = \langle S_N^2(t) \rangle - \langle S_N(t) \rangle^2 \equiv \sigma_N^2(t), \dots$ . If only the first term of the sum in Eq. (3) is kept we arrive at the zeroth-order Rosenstock approximation

$$\Phi_N^{(0)}(t) = e^{\langle S_N(t) \rangle \ln p}. \tag{4}$$

The error made by using this approximation can be estimated by taking into consideration the next exponential term in Eq. (3):  $\Phi_N(t) = \Phi_N^{(0)}(t) [1 + \mathcal{O}(\sigma_N^2 (\ln p)^2)]$ . Thus, the condition  $\sigma_N^2 \ll 1/(\ln p)^2$  must be fulfilled for the zeroth-order Rosenstock approximation to be reasonable. The first-order (extended) Rosenstock approximation is obtained by retaining two terms in the infinite sum of Eq. (3):

$$\Phi_N^{(1)}(t) = \exp \left[ \langle S_N(t) \rangle \ln p + \frac{1}{2} \sigma_N^2 (\ln p)^2 \right]. \tag{5}$$

The relative error of this expression is of order  $\mathcal{O}[\kappa_3 (\ln p)^3]$ .

### 3. One-sided trapping problem with a single random walker

The one-sided trapping problem is quite simple for  $N=1$ . Anyway, we report here the main results for the sake of completeness. From Eqs. (1) and (2) the survival probability of a single random walker is

$$\Phi_1(t) = \int_0^{\infty} dr p^r \frac{d}{dr} \operatorname{erf} \left( \frac{r}{\sqrt{4Dt}} \right) = e^{x^2/4} \operatorname{erfc} \left( \frac{x}{2} \right), \tag{6}$$

where  $x = \sqrt{4Dt} \ln(1/p)$ . For very long times,  $x \rightarrow \infty$ , the asymptotic expansion of the complementary error function [27] allows us to write

$$\Phi_1(t) = \frac{2}{\sqrt{\pi}} \frac{1}{x} \left\{ 1 + \sum_{m=1}^{\infty} (-1)^m 2^m \frac{(2m-1)!!}{x^{2m}} \right\}, \tag{7}$$

where  $(2m - 1)!! = (2m - 1) \cdots 5 \cdot 3 \cdot 1$ . Thus, an asymptotic time regime is reached for  $t \gg 1/(\ln p)^2$  where the survival probability exhibits a power-law decay  $\Phi_1(t) \approx 1/[\sqrt{\pi D} \ln(1/p)]t^{-1/2}$ . This is an *algebraic* fluctuation slowdown corresponding to the Donsker–Varadhan limit.

In order to apply the extended Rosenstock approximation to the single random walker case for small  $x$ , we must evaluate the moments of the one-sided span  $S_1(t)$ :

$$\langle S_1^m(t) \rangle = \int_0^\infty \frac{d\Gamma(t|r)}{dr} r^m dr \tag{8}$$

or, after integrating by parts,

$$\langle S_1^m(t) \rangle = m \int_0^\infty \operatorname{erfc}\left(\frac{r}{\sqrt{4Dt}}\right) r^{m-1} dr = \frac{\Gamma(m + 1/2)}{\sqrt{\pi}} (4Dt)^{m/2} \tag{9}$$

with  $\Gamma(m)$  being the gamma or factorial function and where Eq. (2) has been used. It is now easy to verify that  $\kappa_n = a_n(4Dt)^{n/2}$ ,  $n = 1, 2, \dots$  with  $a_1 = 1/\sqrt{\pi}$ ,  $a_2 = 1/2 - 1/\pi$ ,  $a_3 = (4 - \pi)/(2\pi\sqrt{\pi})$ ,  $a_4 = (2\pi - 6)/\pi^2, \dots$ . Direct substitution of these cumulants into the general expression for the Rosenstock approximation (3) yields

$$\begin{aligned} \Phi_1(t) &= \exp\left\{ \sum_{n=1}^\infty \frac{a_n}{n!} (4Dt)^{n/2} \ln^n p \right\} = \exp\left\{ \sum_{n=1}^\infty \frac{(-1)^n}{n!} a_n x^n \right\} \\ &= \exp\left\{ -\frac{1}{\sqrt{\pi}}x + \frac{\pi - 2}{4\pi}x^2 + \frac{\pi - 4}{12\pi\sqrt{\pi}}x^3 + \frac{\pi - 3}{12\pi^2}x^4 + \mathcal{O}(x^5) \right\}. \end{aligned} \tag{10}$$

Notice that the Rosenstock approximation given by Eq. (10) coincides with the exact result in Eq. (6).

#### 4. One-sided span of a set of random walkers

The objective of this section is twofold: first, we want to obtain rigorous asymptotic expansions for the one-sided span moments

$$\langle S_N^m(t) \rangle = \int_0^\infty \frac{d\Gamma_N(t|r)}{dr} r^m dr \tag{11}$$

for  $m = 1, 2, \dots$  and  $N \gg 1$  independent random walkers, and, second, we want to check the reliability of the obtained asymptotic expressions comparing them with numerical results.

Integrating Eq. (11) by parts, we find

$$\langle S_N^m(t) \rangle = m(4Dt)^{m/2} \int_0^\infty \xi^{m-1} [1 - \operatorname{erf}^N(\xi)] d\xi, \tag{12}$$

where  $\xi = r/\sqrt{4Dt}$ . In order to evaluate this integral for large values of  $N$  it suffices to know  $\Gamma(t|r) = \operatorname{erf}(\xi)$  for large  $\xi$ , namely,  $\operatorname{erf}(\xi) \approx 1 - \pi^{-1/2} \xi^{-1} e^{-\xi^2} (1 - \xi^{-2}/2 + \dots)$ . The asymptotic evaluation for large  $N$  of the integral of Eq. (12) is not an easy task.

Fortunately, if one compares this integral with the one carried out in Refs. [24–26], one realizes that both integrals are formally equivalent. In this way one finds

$$\langle S_N^m \rangle \approx [4Dt \ln(N)]^{m/2} [1 - \Delta(m)], \tag{13}$$

where

$$\Delta(m) = \sum_{n=1}^{\infty} \ln^{-n} N \sum_{j=0}^n s_j^{(n)}(m) \ln^j \ln N. \tag{14}$$

Up to second order ( $n=2$ ) the coefficients  $s_j^{(n)}$  are

$$s_0^{(1)}(m) = -\frac{m\omega}{2}, \tag{15}$$

$$s_1^{(1)}(m) = \frac{m}{4}, \tag{16}$$

$$s_0^{(2)}(m) = \frac{m}{8} \left[ (2-m) \left( \frac{\pi^2}{6} + \omega^2 \right) + 2\omega + 2 \right], \tag{17}$$

$$s_1^{(2)}(m) = \frac{m}{8} [(m-2)\omega - 1], \tag{18}$$

$$s_2^{(2)}(m) = \frac{m}{32} (2-m), \tag{19}$$

where  $\omega = \gamma - \frac{1}{2} \ln \pi = 0.0048507 \dots$  and  $\gamma = 0.577215 \dots$  is Euler’s constant. In particular, the average value of  $S_N(t)$  is given by

$$\begin{aligned} \langle S_N \rangle \approx & [4Dt \ln(N)]^{1/2} \\ & \times \left[ 1 - \frac{\ln \ln N - 2\omega}{4 \ln N} - \frac{\frac{1}{4} \ln^2 \ln N - (1 + \omega) \ln \ln N + \tilde{s}_0^{(2)}(1)}{8 \ln^2 N} + \dots \right] \end{aligned} \tag{20}$$

with  $\tilde{s}_0^{(2)}(1) = \pi^2/6 + \omega^2 + 2\omega + 2 = 3.654659 \dots$ . It is remarkable how close the result obtained in Ref. [19], namely,

$$\langle S_N \rangle \approx [4Dt \ln(N)]^{1/2} \left[ 1 - \frac{\ln \ln N + \ln(4\pi)}{4 \ln N} + \dots \right], \tag{21}$$

is to the rigorous result given by Eq. (20). Notice that the result of Eq. (21) starts to differ from that of Eq. (20) in the first corrective term: in this equation  $\ln(4\pi) = 2.531024 \dots$  plays the role of  $-2\omega = -0.009701 \dots$  in Eq. (20).

In Fig. 1, we compare the values of  $\langle S_N^m \rangle / [4Dt \ln N]^{m/2}$  for  $m = 1, 2, \dots, 7$  obtained by integrating Eq. (12) numerically with those obtained by means of the second-order asymptotic approximation of Eq. (13). The importance of the corrective terms given by  $\Delta$  is evident as well as the good performance (even for not-too-large values of  $N$ ) of the second-order asymptotic expression. This is especially notable for low-order moments. Notice that, at least for  $n = 1, 2$ , the coefficient  $s_j^{(n)}(m)$  is a polynomial of degree  $n$  in  $m$ , and that, if this property holds for all  $n$ , it would explain why the truncated asymptotic expansion of  $\langle S_N^m \rangle$  worsens for increasing values of  $m$ .

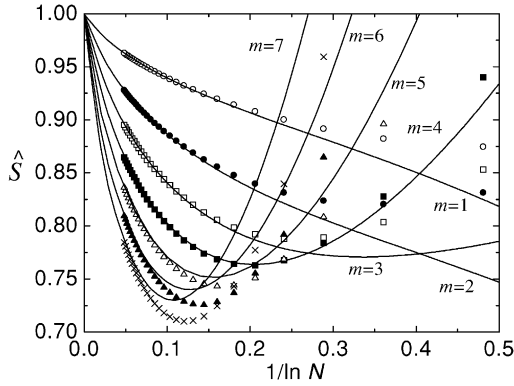


Fig. 1. The dependence on  $N$  of the first seven moments of  $S_N(t)$ . We have plotted  $\hat{S} \equiv \langle S_N^m \rangle / [4Dt \ln N]^{m/2}$  versus  $1/\ln N$ . The symbols correspond to the numerical estimate for  $N = 2^n$  with  $n = 3, 5, \dots, 30$ , when  $m = 1$  (open circles), 2 (filled circles), 3 (open squares), 4 (filled squares), 5 (open triangles), 6 (filled triangles) and 7 (crosses). The solid lines correspond to the second-order asymptotic approximation  $1 - \Delta(m)$  as given by Eq. (14). Notice the importance of the corrective terms as the main term (zeroth-order approximation) would be an horizontal line passing by  $\hat{S} = 1$ .

From Eq. (13) one finds

$$\begin{aligned} \sigma_N(m) &\equiv \langle S_N^m \rangle - \langle S_N \rangle^m \\ &= \frac{m}{48} (m-1) \pi^2 (4Dt)^{m/2} (\ln N)^{m/2-2} \left[ 1 + \mathcal{O} \left( \frac{\ln^3 \ln N}{\ln N} \right) \right], \end{aligned} \tag{22}$$

i.e.,

$$\frac{\sigma_N(m)}{\langle S_N \rangle^m} = \frac{m(m-1)\pi^2}{48 \ln^2 N} \left[ 1 + \mathcal{O} \left( \frac{\ln^3 \ln N}{\ln N} \right) \right]. \tag{23}$$

Fig. 2 is a plot of the numerical and analytical results for  $R(m) \equiv [\langle S_N \rangle^m / \sigma_N(m)]^{1/2}$  versus  $\ln N$ . It is clear that the numerical results closely follow the theoretical prediction given by Eq. (23) although some differences are noticeable. We attribute the difference to the existence of corrective terms of order  $\ln^3 \ln N / \ln N$ .

Let us look at this question more closely. In Fig. 3, we have plotted the ratio

$$\hat{R}(2) \equiv \frac{\sqrt{24}}{\pi} \ln(N) \frac{\sigma_N}{\langle S_N \rangle} \approx 1 + \mathcal{O} \left( \frac{\ln^3 \ln N}{\ln N} \right) \tag{24}$$

evaluated numerically versus  $1/\ln N$ , where  $\sigma_N^2 \equiv \sigma_N(2)$  is the variance. We note that, at first sight, there are two features that cause unease in this figure: first, the numerical results are relatively far from unity (the main asymptotic term) even for very large values of  $N$ , and second, it is difficult to state that unity is the final value for  $N \rightarrow \infty$  by only looking at the points (the numerical results). We can shed some light on these two aspects by considering the form and value of the corrective terms of Eq. (24). The first

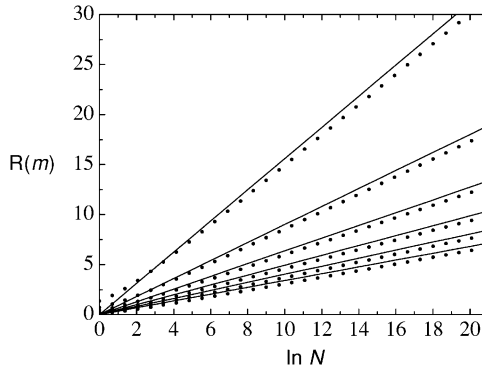


Fig. 2. The ratio  $R(m) \equiv [\langle S_N \rangle / \sigma_N(m)]^{m/2}$  versus  $\ln N$  for (from top to bottom)  $m = 2, 3, \dots, 7$ . The symbols correspond to the numerical evaluation for  $N = 2^n$  with  $n = 0, 1, \dots, 30$ . The lines correspond to the asymptotic result of Eq. (23). The slope of the linear fit to the last five points ( $N$  from  $2^{26}$  to  $2^{30}$ ) is 1.53 (1.56) for  $m = 2$ , 0.88 (0.90) for  $m = 3$ , 0.62 (0.64) for  $m = 4$ , 0.48 (0.49) for  $m = 5$ , 0.39 (0.40) for  $m = 6$ , and 0.33 (0.34) for  $m = 7$ , in good agreement with the corresponding asymptotic value  $\{48/[m(m - 1)\pi^2]\}^{1/2}$  given by Eq. (23) which we have set in parentheses.

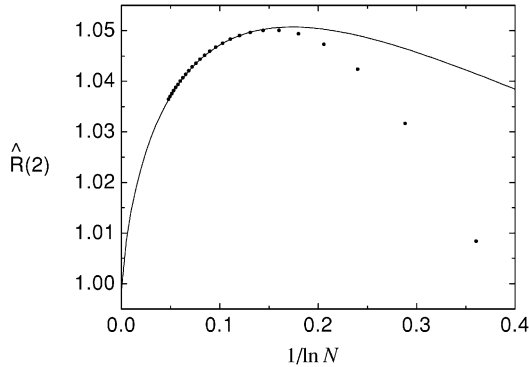


Fig. 3. The ratio  $\hat{R}(2) \equiv (\sqrt{24}/\pi) \ln(N) \sigma_N / \langle S_N \rangle$  versus  $1/\ln N$ . The symbols correspond to the numerical evaluation for  $N = 2^4, 2^5, \dots, 2^{30}$ . The line is a curve of the form  $a(1 + \ln^{-1} N \sum_{j=0}^3 b_j \ln^j \ln N)$  fitted to the last 15 simulation points, i.e., the points corresponding to  $N = 2^{16}, 2^{17}, \dots, 2^{30}$ . The fitting parameters are  $a = 0.998$ ,  $b_0 = 0.009$ ,  $b_1 = -0.019$ ,  $b_2 = 0.094$  and  $b_3 = 0.005$ .

aspect can be understood by taking into account that the functions  $\ln^n(\ln N)/\ln N$  have non-negligible values and are only (slowly) decreasing functions for very large values of  $N$  (e.g., when  $n = 3$ , this function is only decreasing for  $N \gtrsim 5 \times 10^8$ ). In order to resolve the second difficulty, we use our knowledge of the form of the first corrective term of Eq. (24), namely,  $\sum_{j=0}^3 a_j^{(3)} \ln^j \ln(N)/\ln N$ , to fit the numerical results of the quantity  $\sqrt{24}(\ln N) \sigma_N / (\pi \langle S_N \rangle)$  to the expression  $a(1 + \ln^{-1} N \sum_{j=0}^3 b_j \ln^j \ln N)$ . Neglecting, for example, the values corresponding to  $N = 2^0, \dots, 2^{10}$  (recall that our



expressions are asymptotic expressions valid for large  $N$ ), the fitted function leads to  $a \simeq 0.998$ , a result that is in excellent agreement with our predicted value of unity [i.e., the main term of Eq. (24)].

Finally, it is interesting to note that Eq. (22) tells us that, up to *first order* in  $1/\ln N$ , we can get the  $m$ th moment  $\langle S_N^m \rangle$  from only the knowledge of the first moment  $\langle S_N \rangle$ . To be more precise, let us define the parameters  $\hat{s}_j^{(n)}(m)$  through the relationship

$$\langle S_N \rangle^m \approx [4Dt \ln(N)]^{m/2} [1 - \hat{\Delta}(m)], \tag{25}$$

where

$$\hat{\Delta}(m) = \sum_{n=1}^{\infty} \ln^{-n} N \sum_{j=0}^n \hat{s}_j^{(n)}(m) \ln^j \ln N. \tag{26}$$

Hence, using Eqs. (13)–(15), we find (i) that  $s_j^{(n)}(m) = \hat{s}_j^{(n)}(m)$  for  $j = 0, 1$  when  $n = 1$  and for  $j = 1, 2$  when  $n = 2$ , and (ii) that the first terms of  $\langle S_N^m \rangle$  and  $\langle S_N \rangle^m$  that are different are *second-order* terms, namely, those corresponding to  $j = 0$  and  $n = 2$ :  $\hat{s}_0^{(2)}(m) - s_0^{(2)}(m) = m(m - 1)\pi^2/48$ .

It is also worthwhile noting that, working up to second-order asymptotic corrective terms ( $n = 2$ ), the  $j$ th cumulant with  $j > 2$  is zero, or, in other words, that the distribution of  $S_N$  is, up to this second asymptotic order, Gaussian.

### 5. One-sided trapping problem with a set of random walkers

From Eq. (1) and after integrating by parts, we can write the survival probability of a set of  $N$  independent random walkers in the one-dimensional one-sided problem as

$$\Phi_N(t) = x \int_0^{\infty} e^{-x\xi} [\text{erf}(\xi)]^N d\xi, \tag{27}$$

where  $x = \sqrt{4Dt} \ln(1/p)$  and  $\xi = r/\sqrt{4Dt}$ .

The asymptotic behaviour of  $\Phi_N(t)$  for an arbitrary number of particles and  $x \rightarrow \infty$  is a direct consequence of Watson’s lemma [28] and the expansion of the error function  $\text{erf}(\xi)$  for small  $\xi$ :

$$\Phi_N(t) = \left( \frac{2}{\sqrt{\pi}} \right)^N \left[ \frac{N!}{x^N} - \frac{(N + 2)!N}{3x^{N+2}} + \dots \right]. \tag{28}$$

For  $N = 1$  we recover the result in Eq. (7). A slow power-law time decay of  $\Phi_N(t)$  is observed in this limit,  $\Phi_N(t) \approx 2^N N! / (\sqrt{4\pi D} \ln(1/p))^N t^{-N/2}$ .

It is interesting to note that the present one-sided multiparticle Rosenstock trapping problem is related to one of the predator–prey problems discussed by Krapivsky and Redner [18,19] in which a static prey or “lamb” is captured by one of a set of  $N$

diffusing predators or “pride of lions”. These authors considered the case of  $N$  predators and a prey at given relative positions so that the case we study here differs in the sense that the traps (or preys) are randomly distributed. In their analysis, Krapivsky and Redner found that the survival probability of the “lamb” is given by the power law  $t^{-N/2}$ . So, this behaviour agrees with that of our stochastic “lamb” problem in the long-time regime,  $x \gg 1$ , i.e.,  $t \gg \ln N / [\ln(1-c)]^2$ . The reason for this behaviour is that, for very long times, how the traps are configured is not essential in the trapping kinetics, and the slow  $\Phi_N(t) \sim t^{-N/2}$  power-law decay of the fixed tramp (or “lamb”) case settles down.

Next, we will deal with the asymptotic behaviour of  $\Phi_N(t)$  for small  $x$  and large  $N$ . In order to implement the Rosenstock approximation, we will use the expressions for the average one-sided span  $\langle S_N(t) \rangle$  and its variance  $\sigma_N(2) = \sigma_N^2$  given by Eqs. (20) and (22). Then, the zeroth-order Rosenstock approximation given by Eq. (4) is

$$\begin{aligned} \ln \Phi_N^{(0)}(t) &= (4Dt \ln N)^{1/2} \ln(p) \\ &\times \left[ 1 - \frac{\ln \ln N - 2\omega}{4 \ln N} - \frac{\frac{1}{4} \ln^2 \ln N - (1 + \omega) \ln \ln N + \pi^2/6 + \omega^2 + 2\omega + 2}{8 \ln^2 N} \right. \\ &\quad \left. + \mathcal{O} \left( \frac{\ln^3 \ln N}{\ln^3 N} \right) \right]. \end{aligned} \quad (29)$$

In Section 2, it was mentioned that the relative error for  $\Phi_N^{(0)}$  is of order  $\mathcal{O}(\sigma_N^2 \ln^2 p)$ , so that the condition  $\sigma_N^2 \ll 1/(\ln p)^2$  must be fulfilled for the zeroth-order Rosenstock approximation to be reasonable. Using the expression for  $\sigma_N^2$  given by Eq. (22), one sees that the relative error goes as  $\pi^2 Dt / [6(\ln p)^2 \ln N]$ , so that  $\Phi_N^{(0)}(t)$  should give good results when  $t \ll \ln N / (\ln p)^2$  or  $x \ll \sqrt{\ln N}$ . Consequently, there exists a time regime, which becomes *larger* as the number of random walkers *increases* and the concentration of traps decreases, where the approximation for the survival probability in Eq. (29) must work reasonably well. We will denote by  $\Phi_N^{(0m)}(t)$ ,  $m = 0, 1, \dots$ , the zeroth-order Rosenstock approximation for the survival probability that results from retaining  $m$  corrective terms in the expression for the average one-sided span and ignoring the rest of the terms. For example,

$$\Phi_N^{(01)}(t) = \exp \left[ -x (\ln N)^{1/2} \left( 1 - \frac{\ln \ln N - 2\omega}{4 \ln N} \right) \right]. \quad (30)$$

Keeping all the explicit terms in Eq. (29), we get  $\Phi_N^{(02)}(t)$ .

Proceeding as above, the approximation given by Eq. (5) that includes the variance term, i.e., the first-order Rosenstock approximation, is

$$\begin{aligned} \ln \Phi_N^{(1)}(t) &= (4Dt \ln N)^{1/2} \ln(p) \\ &\times \left[ 1 - \frac{\ln \ln N - 2\omega}{4 \ln N} - \frac{\frac{1}{4} \ln^2 \ln N - (1 + \omega) \ln \ln N + \pi^2/6 + \omega^2 + 2\omega + 2}{8 \ln^2 N} \right. \end{aligned}$$

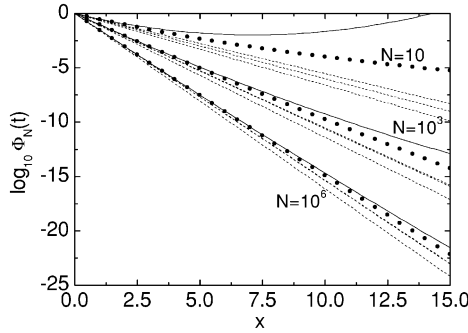


Fig. 4. Survival probability of  $N = 10, 10^3$  and  $10^6$  random walkers in the one-dimensional one-sided Rosenstock trapping model versus  $x = \sqrt{4Dt} \ln(1/p)$ . The circles correspond to a numerical integration of the exact result, the broken lines correspond to the Rosenstock approximations of orders  $\Phi_N^{(00)}, \Phi_N^{(01)}, \Phi_N^{(02)}$  enumerated from below, and the solid line corresponds to  $\Phi_N^{(1)}$ . The lines corresponding to  $\Phi_N^{(01)}$  and  $\Phi_N^{(02)}$  for  $N = 10^6$  are indistinguishable.

$$\begin{aligned}
 & + \mathcal{O} \left( \frac{\ln^3 \ln N}{\ln^3 N} \right) \Bigg] \\
 & + (4Dt \ln N) \ln^2(p) \left[ \frac{\pi^2}{48 \ln^2 N} + \mathcal{O} \left( \frac{\ln^3 \ln N}{\ln^3 N} \right) \right]. \tag{31}
 \end{aligned}$$

We have seen in Section 2 that one can estimate the error of the first-order Rosenstock approximation by looking at the value of  $\kappa_3(\ln p)^3$ . The kurtosis,  $\kappa_3$ , was not calculated in Section 4 explicitly, but we know (see the last paragraph of that section) that it is zero up to at least second-order corrective terms, i.e., at least,  $\kappa_3 = \mathcal{O}(t^{3/2} \ln^3 \ln N \times \ln^{-3/2} N)$ . This means that the relative error of the first-order generalized Rosenstock approximation is a quantity of order  $\mathcal{O}[(x/\sqrt{\ln N})^3 \ln^3 \ln N]$  which will be much smaller than that corresponding to Eq. (4), i.e.,  $\mathcal{O}(x/\sqrt{\ln N})$ , as long as the condition  $x \ll \sqrt{\ln N}$  is fulfilled.

In Fig. 4, we compare the different order Rosenstock approximations with the numerical evaluation of Eq. (27) for  $N = 10, 10^3$  and  $10^6$ . We observe that adding more corrective terms increases the agreement of the Rosenstock approximation with the numerical result for small values of  $x$ . Quite noticeably, the approximations also get better as  $N$  increases.

### 6. Lifetime in the one-sided trapping problem

Let the lifetime of the set of  $N$  independent random walkers  $T_N$  be defined as the time at which some random walker of this set is first trapped or, conversely, the time at which the lamb is killed by the pride of  $N$  lions if the expression coined by Krapivsky

and Redner is used [18,19]. The  $m$ th moment of the lifetime distribution is given by

$$\langle T_N^m \rangle = - \int_0^\infty t^m \frac{d\Phi_N(t)}{dt} dt \tag{32}$$

or, taking into account Eq. (27),

$$\langle T_N^m \rangle = \ln(1/p) \int_0^\infty dr p^r \bar{t}_N^m(r), \tag{33}$$

where  $\bar{t}_N^m(r)$  is the  $m$ th moment of the time to first reach the distance  $r$  by the first random walker of a set  $N$  independent diffusing random walkers:

$$\bar{t}_N^m(r) = - \int_0^\infty t^m \frac{d}{dt} \left[ \operatorname{erf} \left( \frac{r}{\sqrt{4Dt}} \right) \right]^N dt. \tag{34}$$

Of course,  $\bar{t}_N^m(r)$  can be understood as the moments of the lifetime in the trapping problem with a fixed trap at a distance  $r$  from the starting site of  $N$  independent random walkers. This problem has been widely studied. For example, Lindenberg et al. [29] studied the first passage time for small  $N$ , finding that the  $m$ th moment of the first-passage-time distribution for the first of the walkers to reach  $r$  is finite if there are at least  $2m + 1$  random walkers starting from the origin. Hence, by using Eq. (33), we can extrapolate these conclusions to the one-dimensional one-sided Rosenstock’s trapping problem. In particular, this means that  $\langle T_1 \rangle$  and  $\langle T_2 \rangle$  are infinite but  $\langle T_N \rangle = C_N / (2D \ln^2 p)$  is finite for every  $N \geq 3$ , the coefficient  $C_N$  being given by  $\int_0^\infty [\operatorname{erf}(t^{-1/2})]^N dt$ .

The trapping problem with fixed trap and large  $N$  has also been studied [30–33]. In particular, for the one-dimensional lattice [31–33]

$$\bar{t}_N^m(r) = \left( \frac{r^2}{4D} \right)^m \frac{1}{\ln^m \lambda_0 N} \left\{ 1 + \frac{m}{\ln \lambda_0 N} \left( \frac{1}{2} \ln \ln \lambda_0 N - \gamma \right) + \dots \right\}, \tag{35}$$

where  $\lambda_0 = 1/\sqrt{\pi}$ . Then, from Eqs. (33) and (35) we have

$$\langle T_N^m \rangle = \bar{t}_N^m \left( \frac{[(2m)!]^{1/2m}}{\ln 1/p} \right). \tag{36}$$

Thus, the  $m$ th moment of the lifetime  $\langle T_N^m \rangle$  coincides with the  $m$ th moment of the first passage time to a fixed trap at a distance  $[(2m)!]^{1/2m}$  times the average distance  $\langle l \rangle = -1/\ln p$  of a trap to the origin in the one-sided Rosenstock model with trap density  $c = 1 - p$ . According to Eqs. (36) and (35), and writing the expansion in terms of  $\ln N$  instead of  $\ln \lambda_0 N$ , we finally have

$$\begin{aligned} \langle T_N^m \rangle = & \frac{(2m)!}{[4D(\ln p)^2 \ln N]^m} \left\{ 1 + \frac{m}{\ln N} \left( \frac{1}{2} \ln \ln N - \omega \right) + \frac{m}{2 \ln^2 N} \right. \\ & \times \left[ 1 + \frac{(m+1)\pi^2}{6} + (m+1)\omega^2 + \omega - \left( m\omega + \omega + \frac{1}{2} \right) \ln \ln N \right. \\ & \left. \left. + \frac{(m+1)}{4} \ln^2 \ln N \right] + \mathcal{O} \left( \frac{\ln^3 \ln N}{\ln^3 N} \right) \right\}. \end{aligned} \tag{37}$$

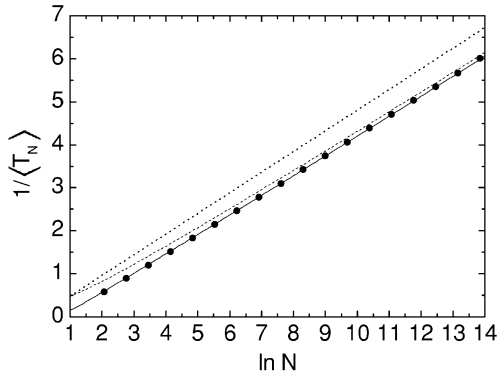


Fig. 5. Inverse of the lifetime of a set of  $N$  independent random walkers in the one-dimensional one-sided Rosenstock trapping model with a density of traps  $c = 1/2$  versus  $\ln N$ . Numerical integration results are plotted as circles and the lines are theoretical asymptotic predictions of zeroth (dotted), first (broken), and second orders (solid). The value  $D = 1/2$  has been used.

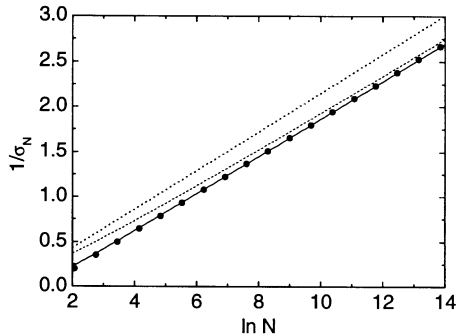


Fig. 6. The same as Fig. 5 but for the inverse of the square root of the variance,  $1/\sigma_N$ .

The asymptotic expansion for the variance of the distribution of first passage times,  $\sigma_N^2 = \langle T_N^2 \rangle - \langle T_N \rangle^2$ , is calculated from Eq. (37):

$$\begin{aligned} \sigma_N = & \frac{\sqrt{20}}{4D(\ln p)^2} \frac{1}{\ln N} \left\{ 1 + \frac{1}{\ln N} \left( \frac{1}{2} \ln \ln N - \omega \right) + \frac{1}{2 \ln^2 N} \right. \\ & \times \left[ 1 + \frac{8\pi^2}{15} + \omega(1 + 2\omega) - \left( \frac{1}{2} + 2\omega \right) \ln \ln N + \frac{1}{2} \ln^2 \ln N \right] \\ & \left. + \mathcal{O} \left( \frac{\ln^3 \ln N}{\ln^3 N} \right) \right\}. \end{aligned} \tag{38}$$

The large size of the variance is remarkable: notice that the coefficient of variation  $\langle T_N \rangle / \sigma_N \approx 1/\sqrt{5}$  (for  $N \gg 1$ ) is less than unity.

In Figs. 5 and 6, we have plotted the inverse of the average first passage time,  $1/\langle T_N \rangle$ , and the inverse of the square root of the variance,  $1/\sigma_N$ , respectively, versus  $\ln N$ . The

second-order approximations for the first passage time in Eq. (37) and the variance in Eq. (38) are in excellent agreement with the results of the numerical integration of Eq. (33).

## 7. Conclusions and remarks

In this paper we have studied the one-dimensional one-sided multiparticle Rosenstock trapping model in which  $N$  independent random walkers start their random exploration at the same site,  $x=0$ , of a one-dimensional lattice whose right side,  $x>0$ , has been randomly filled with static traps with a density  $c \equiv 1 - p$ . This problem can be seen either as a multiparticle version of the usual “two-sided” trapping model, or as “random” version of the “predator–prey” problems such as that discussed in Refs. [18,19] regarding the survival probability of the prey, or in Ref. [29] regarding the prey’s lifetime. Our main interest has been the calculation of the survival probability  $\Phi_N(t)$  and the moments of the lifetime  $\langle T_N^m \rangle$ .

For evaluating  $\Phi_N(t)$  for large  $N$  we resorted to the (extended) Rosenstock approximation. This required, as an intermediate step but interesting in itself, the asymptotic evaluation of the moments of the one-sided span  $S_N(t)$  in terms of a series of the form  $\langle S_N^m \rangle \approx (4Dt \ln N)^{m/2} [1 - \sum_{n=1}^{\infty} (\ln N)^{-n} \sum_{j=0}^n s_j^{(n)} \ln^j \ln N]$ . The importance of evaluating the corrective terms even for very large values of  $N$  is evident as these terms decay mildly as powers of  $1/\ln N$ . It is worth mentioning that we have found that  $\langle S_N^m(t) \rangle = \langle S_N(t) \rangle^m$  up to *first-order* corrective terms ( $n=1$ ). The agreement of the Rosenstock approximation with numerical results for  $\Phi_N(t)$  improves when  $N$  increases and  $x = -\sqrt{4Dt} \ln(1-c)$  decreases, the agreement being very good when  $x \ll \sqrt{\ln N}$ . However, the stretched exponential behaviour of  $\Phi_N(t)$  characteristic of the Rosenstock approximation breaks down for  $x \gg \sqrt{\ln N}$  and an algebraic long-time tail behaviour settles in:  $\Phi_N(t) \sim t^{-N/2}$ . This is the fluctuation slowdown reported in the trapping problem literature [34–37] and known as the Donsker–Varadhan limit, but here more dramatic, being algebraic instead of stretched exponential, because of the infinite half-line free of traps that exists in the one-dimensional one-sided trapping model.

Once the survival probability was determined, we dealt with the problem of the first passage time  $T_N$  of the first random walker of a set of  $N$  to the nearest site occupied by a trap. We found that the  $m$ th moment  $\langle T_N^m \rangle$  coincides with the  $m$ th moment of the first passage time to a single fixed trap placed at the distance  $r_m = [(2m)!]^{1/(2m)} \langle l \rangle$  from the origin,  $\langle l \rangle = 1/\ln(1/p)$  being the average distance to the origin of the nearest trap of the random distribution.

Finally, it must be remarked that the extended Rosenstock approximation could also be used for the multiparticle trapping problem on the “two-sided” one-dimensional lattice and on other Euclidean and fractal substrates because asymptotic series for the average number of distinct sites visited  $\langle \tilde{S}_N(t) \rangle$  have previously been derived for these media [24–26]. However, the calculation of higher order moments of  $\tilde{S}_N(t)$  poses a

problem of completely different order of magnitude that still remains unsolved. On the other hand, we have found that, up to first-order corrective terms,  $\langle S_N^2(t) \rangle = \langle S_N(t) \rangle^2$ . At this point, it is tempting to conjecture that this holds for  $\tilde{S}_N(t)$  too. Were this true, the variance of  $\tilde{S}_N(t)$  would be a quantity of *second* order and, therefore, the zeroth-order Rosenstock approximation with  $\langle \tilde{S}_N \rangle$  given up to first-order corrective term should lead to a good approximation for the survival probability  $\Phi_N(t)$  for Euclidean and fractal media too. Work is in progress to check these conjectures and thereby extend the results of this present work to more general trapping problems.

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