Territory covered by N random walkers on fractal media: The Sierpinski gasket and the percolation aggregate

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We address the problem of evaluating the number $S_N(t)$ of distinct sites visited up to time t by N noninteracting random walkers all starting from the same origin in fractal media. For a wide class of fractals (of which the percolation cluster at criticality and the Sierpinski gasket are typical examples) we propose, for large N and after the short-time compact regime, an asymptotic series for $S_N(t)$ analogous to that found for Euclidean media: $S_N(t) \approx \hat{S}_N(t)(1-\Delta)$. Here $\hat{S}_N(t)$ is the number of sites (volume) inside a hypersphere of radius $L[\ln(N)/c]^{1/v}$ where L is the root-mean-square chemical displacement of a single random walker, and v and c determine how fast $1-\Gamma_t(\ell)$ (the probability that a given site at chemical distance ℓ from the origin is visited by a single random walker by time t) decays for large values of ℓ/L: $1-\Gamma_t(\ell) \sim \exp[-c(\ell/L)^v]$. For the fractals considered in this paper, $v = d_w^\ell/(d_w^\ell - 1)$, d_w^ℓ being the chemical-diffusion exponent. The corrective term Δ is expressed as a series in $\ln^{-n}(N)\ln^m \ln(N)$ (with $n \geq 1$ and $0 \leq m \leq n$), which is given explicitly up to $n = 2$. This corrective term contributes substantially to the final value of $S_N(t)$ even for relatively large values of N.

I. INTRODUCTION

Random walk theory is a branch of statistical physics with many applications [1,2]. Problems related to a single random walker have traditionally been the subject of thorough study, but their generalizations to $N > 1$ random walkers have attracted much less attention, although there are some, generally very recent, exceptions [3–8]. These multiparticle diffusion problems are characterized by the impossibility of being analyzed in terms of the single random walker theory, i.e., they cannot be solved through simple averaging over the properties of a single random walker, even in the noninteracting case. The recent development of experimental techniques allowing the observation of events caused by single particles of an ensemble [9] should give additional encouragement to the study of these multiparticle diffusion problems.

The subject of this paper, namely, the evaluation of the average number $S_N(t)$ of distinct sites visited (or territory covered) by N random walkers up to time t, all moving from the same starting site, is a clear example of a diffusion problem that cannot be solved, or even approximated, from the solution for $N = 1$, $S_1(t)$. Even for independent random walkers, the overlap of the regions explored by different walkers prohibits a decomposition of $S_N(t)$ into single-particle contributions. The origin of the problem of evaluating $S_N(t)$ is usually traced back to the year 1951 when the case $N = 1$ was posed by Dvoretzky and Erdös [10]. Since then, the quantity $S_1(t)$ has been studied in detail and is discussed in general references [1]. For fractal substrates this problem was studied by Rammal and co-workers [11]. More recently, Larralde *et al.* [3] and Havlin *et al.* [4] studied the problem of evaluating $S_N(t)$ when $N \gg 1$ noninteracting random walkers diffuse in Euclidean and fractal media, respectively. For fractal lattices with spectral dimension $d_s = 2d_f/d_w < 2$, it was argued by Havlin *et al.* [4] that $S_N(t)$

$\sim t^{d_\ell}$ for $t < t \ll t_\times \sim \ln N$ and $S_N(t) \sim t^{d_s/2}(\ln N)^{d_f/u}$ for $t_\times \ll t$, where $d_\ell = d_f/d_{\min}$ is the chemical dimension (or topological distance dimension), $d_{\min}$ is the fractal dimension of the shortest path on the fractal, d_f is the fractal dimension, $u = d_w/(d_w - 1)$, and d_w is the diffusion exponent (or fractal dimension of the random walk) [2,12]. However, Dräger and Klafter [8] using scaling arguments have recently proposed that

$$S_N(t) \sim t^{d_s/2}(\ln N)^{d_\ell/v} \tag{1}$$

for $t_\times \ll t$, where $v = d_w^\ell/(d_w^\ell - 1)$ and $d_w^\ell = d_w/d_{\min}$ is the chemical-diffusion exponent [2,12,13]. Of course, the two predictions agree for those media, such as Sierpinski gaskets, for which $d_{\min} = 1$.

As stated above, two time regimes are observed in $S_N(t)$: an extremely short-time regime or regime I and a long-time regime or regime II separated by the crossover time $t_\times \sim \ln N$. A further long-time regime, or regime III, is observed in Euclidean lattices when the movement of the independent walkers are very far from each other so that their trails (almost) never overlap and $S_N(t) \sim NS_1(t)$ [3,7]. In the one-dimensional lattice and fractal lattices with $d_s \leq 2$, the trails of the random walkers partially overlap at all times and regime III is never reached. Such is the case in this paper where we are concerned only with fractals in which $d_s < 2$. Regime I and its transition to regime II is well understood [3,4] and it will not be discussed here.

Regime II is far more interesting and difficult to analyze than regime I due to the nontrivial interplay of the walkers in their exploration of the lattice. In some recent work [7], we have shown that for independent random walks on Euclidean lattices there exist important asymptotic corrections to the main term of $S_N(t)$ that cannot be ignored even for a very large number of particles as these corrections decay only logarithmically with N. We will see that this also holds for the fractal lattices considered in this paper. An important

consequence that we will address in Sec. V B is that the corrective terms must be taken into account in the analyses based on ''collapsing'' the numerical data [3,4,8] to determine the exponents in the main term of $S_N(t)$.

It should be noticed that, except for the Sierpinski gasket in two dimensions when $N=1$ [11], there has never been any discussion about $S_N(t)$ focused on deterministic fractals, whether theoretically or numerically. Certainly, a dependence on t and N of the main asymptotic term of $S_N(t)$ for large N has been proposed [see Eq. (1)], but nothing is known about the value of its amplitude or prefactor and on the relevance (if any) of the other (corrective) asymptotic terms. In this paper we present a procedure for obtaining, for a certain class of fractals, the complete asymptotic series expansion of $S_N(t)$ when $N \gg 1$. The procedure gives the main asymptotic term in full, and determines the functional form of the corrective terms, which we calculate explicitly up to second order.

Stochastic (or disordered) fractal media are not constructed by the iteration of an invariable rule, but are rather the result of a random process. Their fractal nature is recognized by the scaling of statistical quantities. Many natural objects share this statistical-fractal structure [2,12–14] so that stochastic models seem to be more suitable to represent diffusion in real media. The quantity we are interested in, $S_N(t)$, is, for disordered media, the result of a double average: an average over the walks that the N random walkers can perform over a given lattice, followed by an average over many (ideally, all) realizations of the random lattice. This fact leads to certain subtleties, absent in deterministic fractals, that require special treatment and that, at the end, are the cause of the discrepancy between Refs. [4] and [8] that we have already mentioned [see above Eq. (1)].

The paper is organized as follows. A discussion on the relation between the territory covered by N random walkers and the statistical quantity known as survival probability is given in Sec. II. The asymptotic evaluation of $S_N(t)$ on fractal lattices is presented in Sec. III. The mathematical techniques involved are very similar to those corresponding to the Euclidean case and we will only outline the main steps. Details may be found in Ref. [7]. A less rigorous but fairly simple method for obtaining the main asymptotic term and estimating the corrective terms of $S_N(t)$ for large N is also presented. In Sec. IV we compare the asymptotic expansion of $S_N(t)$ with simulation results obtained on the Sierpinski gasket. In Sec. V we report simulation results for the survival probability of a random walker on a two-dimensional incipient percolation aggregate when a trap is placed at a site at a fixed chemical distance ℓ or Euclidean distance r. We find that the distribution is narrow [broad] if the traps are located at a fixed chemical [Euclidean] distance. The parameters governing the asymptotic behavior of the survival probability, how the fractal volume grows, and how fast a single walker diffuses are estimated in this section. We compare the zeroth- and first-order asymptotic expansion for $S_N(t)$ with simulation results obtained for the two-dimensional incipient percolation aggregate. In Sec. VI we end with some remarks on the quality of the asymptotic approximation.

II. SURVIVAL PROBABILITY AND TERRITORY COVERED BY N RANDOM WALKERS

Let $\Gamma_t(\mathbf{r})$ be the probability (survival probability) that a site $\mathbf{r}$ has not been visited by a single random walker by time t. It is well known that $S_N(t)$ can be expressed by [3,4]

$$S_N(t) = \left\langle \sum \left\{ 1 - [\Gamma_t(\mathbf{r})]^N \right\} \right\rangle, \qquad (2)$$

where the sum is over all the sites of each fractal lattice, $\Sigma \{ 1 - [\Gamma_t(\mathbf{r})]^N \}$ represents the mean territory explored by the N random walkers on a given lattice (the first average) [4], and $\langle [\cdots] \rangle$ indicates that the average (the second average) of $[\cdots]$ has to be performed over all possible stochastic lattices compatible with the random generation rules. Equation (2) can be rewritten as

$$S_N(t) = \sum_{m=0}^{\infty} \left\langle \sum_{i=1}^{n(m)} \left\{ 1 - [\Gamma_t(\mathbf{r}_{m,i})]^N \right\} \right\rangle, \qquad (3)$$

where $\mathbf{r}_{m,i}$ stands for the ith site out of $n(m)$ that are separated from the origin by a Euclidean distance between $m\Delta r \equiv r_m$ and $(m+1)\Delta r$ with Δr small (say, of the order of the lattice spacing). If $\Gamma_t(\mathbf{r}_{m,i})$ is almost independent of i and the lattice realization, i.e., if the fluctuations in the probability density $\Gamma_t(\mathbf{r}_{m,i})$ follow a narrow distribution, then one could approximate $\Gamma_t(\mathbf{r}_{m,i}) \simeq \langle \Gamma_t(\mathbf{r}_{m,i}) \rangle \equiv \Gamma_t(r_m)$, and therefore estimate $S_N(t)$ by

$$S_N(t) = \sum_{m=0}^{\infty} \left\{ 1 - [\Gamma_t(r_m)]^N \right\} \langle n(m) \rangle, \qquad (4)$$

where $\langle n(m) \rangle$ is the average number of fractal sites separated from the origin by a distance bracketed by r_m and $r_m + \Delta r$. This is essentially the starting relationship used (implicitly) by Havlin *et al.* [4] to find that, for large N,

$$S_N(t) \sim t^{d_s/2} (\ln N)^{d_f/u} \qquad (5)$$

in the nontrivial time regime (or regime II) with $u = d_w/(d_w - 1)$. However, the hypothesis leading to Eq. (4) on the narrowness of the distribution of $\Gamma_t(\mathbf{r}_{m,i})$ is in general false, as we will explicitly show in Sec. V A by means of numerical simulations for the two-dimensional percolation cluster at criticality. Indeed, it is known [15] that the fluctuations of the probability density $P(r,t)$ of random walks (also called the propagator or Green's function), which is a statistical quantity *closely related* to the survival probability, exhibits a broad logarithmic distribution for some random fractals such as percolation clusters and self-avoiding walks. Bunde *et al.* [15] have found that the quantity $\langle P(r,t)^q \rangle$ exhibits multifractal scaling, $\langle P(r,t)^q \rangle \sim \langle P(r,t) \rangle^{\tau(q)}$, where $\tau(q) \sim q^\gamma$ and $\gamma = (d_w^\ell - 1)/(d_w - 1)$. This behavior is a consequence of the large fluctuations of $P(r,t)$ for fixed r and t from a given aggregate to another. Nevertheless, these authors have also shown that the distribution of the propagator in the chemical ℓ space, $P(\ell,t)$, is narrow and, conse-

quently, $\langle P(\ell,t)^q\rangle \sim \langle P(\ell,t)\rangle^q$ (the chemical distance ℓ is the minimum path length between two sites along lattice bonds on a lattice).

Let $\ell_{m,i}$ label the ith site out of those $n(m)$ that are placed at a chemical distance ℓ from a given origin with $\ell_m \leq \ell < \ell_{m+1}$, $\ell_m = m\Delta\ell$ and $\Delta\ell$ small (say, of the order of the lattice spacing), and let $\Gamma_t(\ell_{m,i})$ be the survival probability in the chemical space defined as the probability that site $\ell_{m,i}$ has not been visited by time t by a single random walker starting from the origin. Then we can rewrite Eq. (3) in the chemical ℓ space as

$$S_N(t) = \sum_{m=0}^{\infty} \left\langle \sum_{i=1}^{n(m)} \{1 - [\Gamma_t(\ell_{m,i})]^N\} \right\rangle. \qquad (6)$$

One may expect that the distribution of $\Gamma_t(\ell_{m,i})$ for fixed ℓ_m and t is as narrow as the distribution of the propagator in the chemical space. (This is indeed the case for the two-dimensional percolation clusters at criticality; see Sec. V A). Therefore

$$\Gamma_t(\ell_{m,i}) \simeq \langle \Gamma_t(\ell_{m,i})\rangle \equiv \Gamma_t(\ell_m) \qquad (7)$$

for all possible lattice realizations so that $\langle [\Gamma_t(\ell_{m,i})]^N\rangle \simeq \langle \Gamma_t(\ell_{m,i})\rangle^N$ and $S_N(t)$ can be approximated by

$$S_N(t) = \sum_{m=0}^{\infty} \{1 - [\Gamma_t(\ell_m)]^N\}\langle n(m)\rangle, \qquad (8)$$

where $\langle n(m)\rangle$ is the average number of fractal sites separated from the origin by a chemical distance with value between ℓ_m and $\ell_m + \Delta\ell$. From this formula and following the procedure in Ref. [7], in Sec. III we will arrive at an expression for $S_N(t)$ for the nontrivial time regime whose leading asymptotic behavior coincides, apart from the value of the prefactor, with the recent proposal, Eq. (1), of Dräger and Klafter [8]. Equation (1) differs from the relationship proposed by Havlin *et al.* [4], Eq. (5), for those media where $d_{min} \neq 1$. Both Havlin *et al.* and Dräger and Klafter supported their conjectures by means of data collapsing plots of computer simulation results obtained for two- and three-dimensional percolation aggregates, respectively. In Sec. V B we will draw attention to the risk involved in this method of analysis when the influence of the corrective terms is not properly considered since these terms have a large influence [16] on the final value of $S_N(t)$.

III. TERRITORY COVERED BY N RANDOM WALKERS ON A STOCHASTIC FRACTAL SUBSTRATE

The fractals that we consider in this paper have to satisfy two conditions. First, the number of sites (or volume) $V(\ell)$ of the fractal inside a hypersphere of chemical radius ℓ should be given by

$$V(\ell) = V_0\ell^{d_\ell}, \qquad (9)$$

where V_0 is a constant characteristic of the fractal substrate; and, second, the probability $\Gamma_t(\mathbf{r})$ that a site $\mathbf{r}$ has not been visited by a single random walker by time t should decay for $\xi \equiv \ell(\mathbf{r})/L \gg 1$ as

$$\Gamma_t(\mathbf{r}) \approx 1 - A\xi^{-\mu v}\exp(-c\xi^v)(1 + h_1\xi^{-v} + \cdots), \quad (10)$$

where $\ell(\mathbf{r})$ is the chemical distance between site $\mathbf{r}$ and the starting site of the random walker, and

$$L^2 \equiv \langle \ell^2\rangle = 2D_\ell t^{2/d_w^\ell} \qquad (11)$$

is the mean-square chemical distance traveled by a single random walker by time t (t large), D_ℓ being the diffusion constant. Equation (11) is known as the Einstein relation. For fractals with $d_{min} = 1$, we will define ξ as $\xi \equiv |\mathbf{r}|/R \equiv r/R$ where $R^2 = 2Dt^{2/d_w}$ is the mean-square Euclidean distance traveled by a single random walker, D is the diffusion constant, and r is the Euclidean distance between site $\mathbf{r}$ and the starting site of the random walker. Equation (10) holds on Euclidean lattices [3,7]. Notice, also, that the dominant asymptotic behavior of the propagator in chemical space [2,12,15], $P(l,t) \sim \exp(-c\xi^v)$, also coincides with the assumed dominant exponential decay of the mortality function $1 - \Gamma_t(\ell)$ in Eq. (10). As the propagator and the mortality function share the same asymptotic behavior for Euclidean lattices and for the Sierpinski lattice [17], we can expect that this behavior also is the case for stochastic fractals (we will check this supposition in Sec. V).

It should be clear at this point that for deterministic fractals the above two conditions can only be satisfied approximately: first, because V_0 is not strictly constant (it exhibits log-periodic oscillations of small amplitude, see Sec. IV); second, because $\Gamma_t(\mathbf{r})$ does not solely depend on the distance r but also (in general) on the actual location $\mathbf{r}$ on the lattice; and, third, because $\Gamma_t(\mathbf{r})$ is not continuous (this fact can be clearly seen in Fig. 1 of Ref. [18]) so that Eq. (10) can only be an approximation to the true distribution. The fluctuations in $S_N(t)$ associated with these effects are thus not included in our theoretical discussion. However, their importance can be gauged by resorting to simulation. For the two-dimensional Sierpinski gasket, we found that these fluctuations are indeed relevant and that they can be explained to a large extent as a consequence of the log-periodic oscillations of V_0.

There is another difficulty regarding the value of $\Gamma_t(\mathbf{r})$ for deterministic and stochastic fractals: while its dominant term $\exp(-c\xi^v)$ is reasonably well established, the value (and even the form) of its subdominant factors $A\xi^{-\mu v}$, $h_1\xi^{-v}$, etc. is unknown. This means that we can be fairly sure of the value of the main term of $S_N(t)$ because, as we will show, it depends only on the dominant term of $\Gamma_t(\mathbf{r})$. However, the true value of the corrective terms of $S_N(t)$ is more uncertain as they also depend on the subdominant factors of $\Gamma_t(\mathbf{r})$. Nevertheless, we will see in Secs. IV and V that reasonable choices of values for these subdominant factors lead to significant improvements in the estimate of $S_N(t)$.

The evaluation of $S_N(t)$ starts by replacing Eq. (8) by its continuum approximation

$$S_N(t) = \int_0^\infty \{1 - [\Gamma_t(\ell)]^N\} d_\ell V_0 \ell^{d_\ell - 1} \, d\ell$$

$$= V_0 d_\ell (2D_\ell)^{d_\ell/2} t^{d_\ell/d_w} \int_0^\infty \{1 - [\Gamma_t(\xi)]^N\} \xi^{d_\ell - 1} \, d\xi, \tag{12}$$

where $dV(\ell) = V_0 d_\ell \ell^{d_\ell - 1} d\ell$ is the average number of fractal sites placed at a chemical distance between ℓ and $\ell + d\ell$, and $\xi \equiv \ell/(\sqrt{2D_\ell} t^{1/d_w})$. The rest of the analysis is identical with that carried out for Euclidean lattices [7]. The result for $S_N(t)$ is

$$S_N(t) \approx \hat{S}_N(t) \left[1 - \frac{d_\ell}{v} \sum_{n=1}^\infty (\ln N)^{-n} \sum_{m=0}^n s_m^{(n)} (\ln \ln N)^m \right] \tag{13}$$

with

$$\hat{S}_N(t) = V_0 (2D_\ell)^{d_\ell/2} t^{d_\ell/d_w} \left(\frac{\ln N}{c} \right)^{d_\ell/v} \tag{14}$$

and

$$s_0^{(1)} = -\omega, \tag{15}$$

$$s_1^{(1)} = \mu, \tag{16}$$

$$s_0^{(2)} = -(\beta - 1) \left(\frac{\pi^2}{12} + \frac{\omega^2}{2} \right) - (ch_1 - \mu\omega), \tag{17}$$

$$s_1^{(2)} = -\mu^2 + (\beta - 1)\mu\omega, \tag{18}$$

$$s_2^{(2)} = -\frac{1}{2}(\beta - 1)\mu^2. \tag{19}$$

Here $\omega = \gamma + \ln A + \mu \ln c$, $\gamma \approx 0.577\,215$ is the Euler constant, and $\beta = d_\ell/v$. The dependence on t and N of the main term of $S_N(t)$ as given by Eq. (14), i.e., $S_N(t) \sim t^{d_s/2} (\ln N)^{d_\ell/v}$, coincides with the prediction of Ref. [8].

A simpler way to estimate the territory covered

We finish this section by showing how to find the full main term of Eq. (13) and even predict the form of the corrective terms by only resorting to extremely simple arguments already used in Ref. [6]. The crucial point is that, for a fixed time t, $1 - [\Gamma_t(\ell)]^N$ approaches a unit step function $\Theta(\ell - \ell_\times)$ when $N \to \infty$, $\ell_\times$ being the step's width (see Fig. 1). The reason for this behavior is clear: For large N, $[\Gamma_t(\ell)]^N$ is only non-negligible when $\Gamma_t(\ell)$ is very close to 1. Obviously this occurs when the root-mean square chemical distance $L(t)$ traveled by the single random walker by time t is small compared with ℓ, i.e., when $\xi = \ell/L(t)$ is large. This in turn implies that in the evaluation of $S_N(t)$ only the behavior of $\Gamma_t(\ell)$ for large ξ is relevant. Then, as $1 - [\Gamma_t(\ell)]^N$ approaches a step function of width $\ell_\times$, the integration of Eq. (12) yields
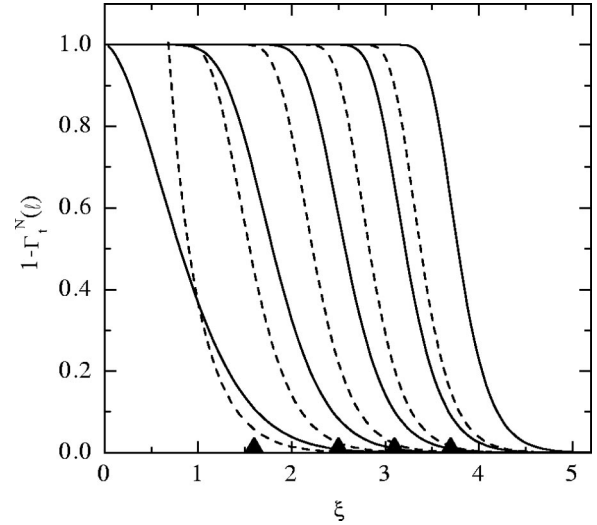

FIG. 1. Function $1 - [\Gamma_t(\ell)]^N$ versus $\xi = \ell/L(t)$ in the two-dimensional incipient percolation aggregate for (from left to right) $N = 1$, 10, 100, 1000, and 10 000 where $\Gamma_t(\ell) = 1 - \xi^{-\mu v} \exp(-\xi^v)$, $c = 1.0$, $v = 1.7$, $\mu = 0$ (solid line), and $\mu = 0.8$ (dashed line). We have not plotted the unphysical values that appear in the case with $\mu = 0.8$ when ξ goes to zero. Notice the large influence of the subdominant power term $\xi^{-\mu v}$ on the value of $[\Gamma_t(\ell)]^N$ which will be reflected in the value of $S_N(t)$. The triangles mark the value of $\xi_\times$ for (from left to right) $N = 10$, 100, 1000, and 10000 obtained from Eq. (22).

$$S_N(t) \approx V_0 \ell_\times^{d_\ell}, \tag{20}$$

i.e., the territory covered is just the volume of a chemical hypersphere of radius $\ell_\times$. Defining the width $\ell_\times$ of the step function as the distance at which $1 - [\Gamma_t(\ell)]^N$ takes the intermediate value 1/2 (any other value between 0 and 1 would also be valid as $\ell_\times$ is not very sensitive to this value when $N \gg 1$), and assuming that $1 - \Gamma_t(\ell) \approx A \xi^{-\mu v} \exp(-c\xi^v)$ for large ξ, we deduce that $1/2 \approx NA\xi_\times^{-\mu v} \exp(-c\xi_\times^v)$, with $\xi_\times \equiv \ell_\times/L$ so that

$$c\xi_\times^v \approx \ln N - \mu v \ln \xi_\times + \ln 2A. \tag{21}$$

The term $\ln N$ is dominant on the right-hand side of Eq. (21) for large N, so that a first-order solution of this equation is

$$c\xi_\times^v \approx \ln N, \tag{22}$$

i.e., $\ell_\times^v \approx L^v \ln(N)/c$. Hence Eq. (20) yields $S_N(t) \approx V_0 L^{d_\ell} (\ln N/c)^{d_\ell/v}$ which is in full agreement with the main term of Eq. (13) when the Einstein relation, Eq. (11), is considered. Inserting the above first-order solution for $\ell_\times$ into the right-hand side of Eq. (21), we get the improved solution $c\xi_\times^v \approx \ln N - \mu \ln \ln N + \ln Ac^\mu + \ln 2$, so that Eq. (20) becomes

$$S_N(t) \approx V_0 L^{d_\ell} \left(\frac{\ln N}{c} \right)^{d_\ell/v}$$

$$\times \left(1 + \frac{d_\ell}{v} \frac{-\mu \ln \ln N + \ln Ac^\mu + \ln 2}{\ln N} \right). \tag{23}$$
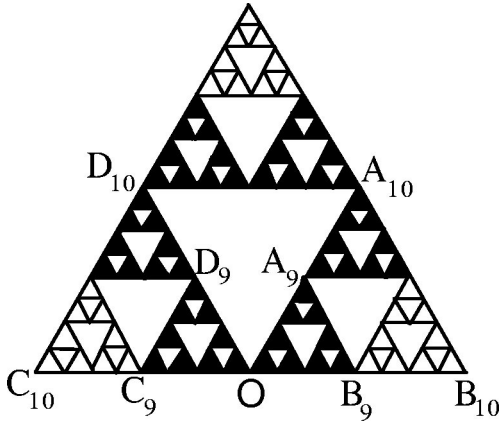
FIG. 2. The 11-generation Sierpinski lattice used in the numerical simulations where the smallest triangles represent eight-generation lattices. The sites labeled A_i, B_i, C_i, and D_i are nearest neighbors of O in the ith times decimated lattice [19,20]. In case (i) all random walkers start from the origin O. In case (ii) the common origin is chosen randomly from among the sites in the shaded area.

This expression is strikingly close to the first-order approximation of Eq. (13), the only difference being that the term $\ln 2 = 0.693 \ldots$ in Eq. (23) plays the role of the Euler constant $\gamma \simeq 0.577215$ in Eq. (13). Finally, notice that this simple method is not limited to fractal media but that it is also valid for estimating $S_N(t)$ for the Euclidean media considered in Ref. [7].

IV. NUMERICAL RESULTS FOR THE SIERPINSKI GASKET

To check the reliability of the analysis presented in the preceding section we carried out simulations of the number of distinct sites visited by N random walkers on a two-dimensional Sierpinski lattice with $g = 11$ generations. This means that if we take the length of any side of the smallest triangles (the zeroth decimated triangles) as the unit length, then the length of the sides of the triangle that inscribes the lattice (the gth decimated triangle) is 2^g. Two different cases are analyzed: (i) random walkers are initially placed upon the center of the base of the main triangle which inscribes the lattice (point O in Fig. 2), and (ii) the common starting site is randomly selected. Qualitatively and quantitatively, the results are different in case (i) and case (ii). The structure of the lattice gives rise to oscillations superimposed on the general trend of $S_N(t)$ in case (i). This structure is smeared out in case (ii) by the double average over experiments and over starting sites, so that $S_N(t)$ is now a smooth function.

A. Case (i)

First, we will discuss the simulation results for the territory covered by N random walkers placed initially at site O in Fig. 2. In order to compare with the zeroth- and first-order asymptotic expression, cf. Eq. (13), we must know the values of V_0, D, c, u, A, and μ. In Fig. 3(a) we have plotted the fractal volume of a circle of radius r, $V(r)$, centered upon the privileged site O. The observed log-periodic structure is a
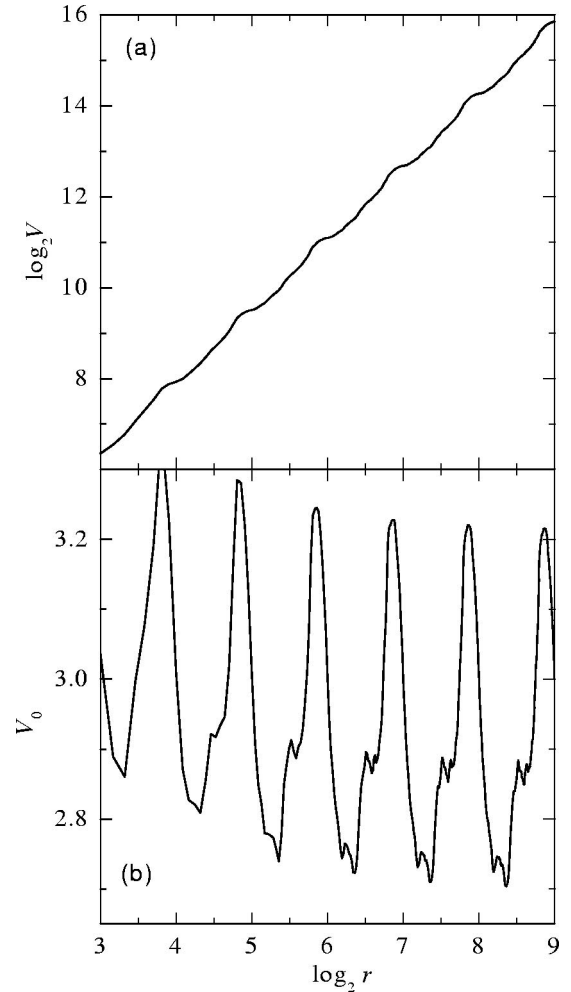

FIG. 3. The fractal volume $V(r)$ of the Sierpinski lattice as a function of r for case (i). (a) $\log_2 V(r)$ versus $\log_2(r)$; (b) $V_0 = V(r)/r^{d_f}$ versus $\log_2(r)$.

consequence of the empty and filled triangular areas that repeat periodically as r is increased, but the general trend is well represented by a term of the form $V_0 r^{d_f}$ with $V_0 \simeq 3.0 \pm 0.1$. In Fig. 3(b), in which the quotient $V(r)/r^{d_f}$ is plotted versus $\log_2 r$, one clearly sees the log-periodic oscillations of V_0. As our theory assumes a constant value for V_0, we take the average value over the last period (from maximum to maximum), $V_0 = 2.93$, as a reasonable criterion for comparison with the simulation results. In order to find the diffusion coefficient D of a random walker starting at O, we performed 10^6 simulations up to $t = 400$. The linear numerical fit between $t = 50$ and $t = 400$ gives $d_w \simeq 2.32$ (the exact value is $d_w = \ln 5/\ln 2 \simeq 2.322$) and $2D \simeq 1.05$. Numerical fits using other time intervals (excluding short times, of course) lead to similar values, and we take $2D \simeq 1.05 \pm 0.02$ as a reliable estimate. For the parameters c and u we take the values 0.981 [17] and $d_w/(d_w - 1) = 1.756$, respectively. As discussed in Sec. III, the values of μ and A are much less certain and we will use here two pairs of values: those obtained by renormalization [19,20], i.e., $\mu = 1/2$ and $A = 0.61$, and these same values increased by one, i.e., $\mu = 3/2$ and $A = 1.61$. Of course, this last pair of parameters are arbitrary (other values
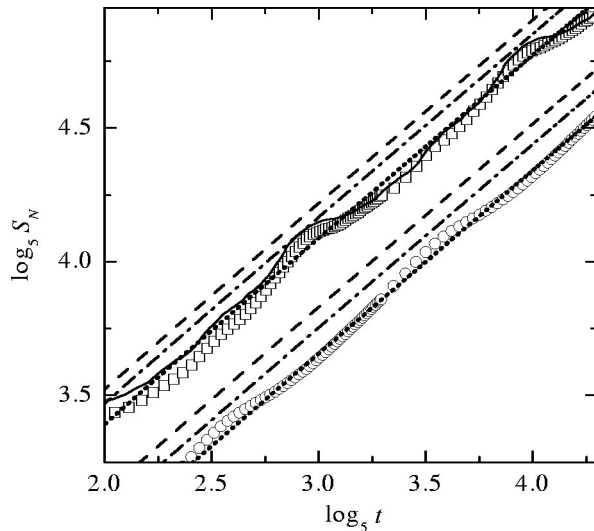
FIG. 4. The number of distinct sites visited on the Sierpinski lattice until $t=1000$ by $N=10^3$ (circles) and $N=10^6$ (squares) random walkers. All random walkers start from the origin O shown in Fig. 2. The dashed line is the zeroth-order approximation, the dot-dashed line corresponds to the first-order approximation with $\mu=1/2$ and $A=0.61$, and the dotted line is the first-order approximation with $\mu=3/2$ and $A=1.61$. The solid line also corresponds to this last approximation but using for V_0 the numerical values of the last oscillation shown in Fig. 3(b).

FIG. 5. The ratio $\mathcal{R}$ between theoretical and numerical values of $S_N(t)$ for the two-dimensional Sierpinski lattice with $N=10^3$ (dashed line) and $N=10^6$ (solid line). The asymptotic approximations considered are (from bottom to top) the zeroth-order approximation, the first-order approximation with $\mu=1/2$ and $A=0.61$, and the first-order approximation with $\mu=3/2$ and $A=1.61$.

could also be used) and are mainly given to show the relevance of the corrective terms.

Simulation results for $S_N(t)$ until $t=1000$ are shown in Fig. 4 for $N=10^3$ and $N=10^6$. Overall good agreement is obtained in the comparison with the theoretical prediction of Eq. (13), especially when the values $\mu=3/2$ and $A=1.61$ are used. This last finding should only be taken as the manifestation of the importance of the corrective terms which can lead to such dramatic changes and improvements in $S_N(t)$ after modifying some of the subdominant factors of the survival probability. Of course, additional independent study will be necessary to check the form of $\Gamma_t(r)$ given by Eq. (10) and to find out whether the values for μ and A considered here are good estimates of the real values [21]. Nevertheless, it should be noted that the decrease in $1-\Gamma_t(r)$ when averaging over the whole lattice with respect to its renormalization value, as is implied by the corresponding increment of μ ($\mu=1/2 \rightarrow \mu=3/2$), is analogous to the decrease of the propagator when this same averaging is performed. Given that the two statistical quantities (propagator and survival probability) are closely related, one is inclined to accept that, at least, the proposed increment in the value of μ captures the right tendency. The subsequent improvement in the prediction of $S_N(t)$ supports this supposition.

The theoretical expression was not able to give a perfect account of the log-periodic oscillations superimposed on the general trend of $S_N(t)$ shown in Fig. 4. The origin of these oscillations is clear: the enlargement of the sides of the Sierpinski gasket by a factor 2 implies that its fractal volume increases by a factor 3 and the time that a random walker takes to traverse it increases by 5 [2].
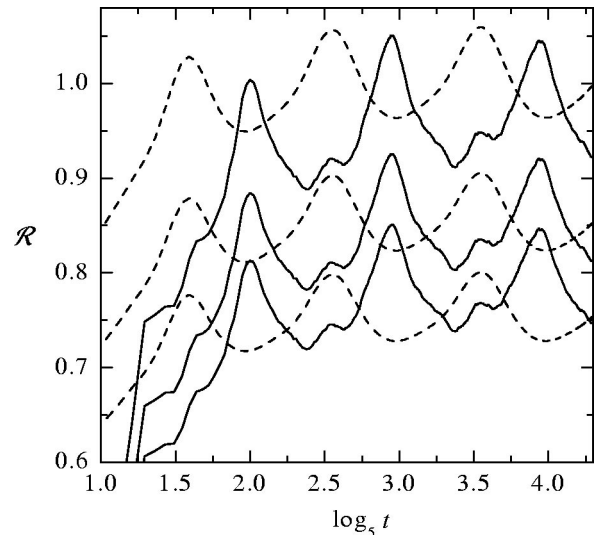
The structure of $S_N(t)$ is more clearly perceived in Fig. 5 where the quotient between the theoretical prediction and the simulation results is plotted. It is remarkable how relatively poor the performance of the zeroth-order approximation (or main asymptotic term) is in predicting the value of $S_N(t)$: it accounts for hardly 80% of $S_N(t)$ for values of N as large as 10^6. However, the inclusion of the first corrective asymptotic term (especially for some suitable selections of the subdominant parameters A and μ) leads to a noticeable improvement. The log-periodic structure is observed both for $N=10^3$ and $N=10^6$ but in the latter case this structure is richer and strikingly similar to that of $V(r)$ as shown in Fig. 3(b). We attribute this fact to a better mapping of the lattice structure as more and more random walkers are involved in the exploration. We have plotted the solid line in Fig. 4 with the aim of showing to what extent the oscillatory behavior of $S_N(t)$ as shown in Fig. 5 can be interpreted as a consequence of the oscillatory behavior of V_0 shown in Fig. 3(b). The line is generated in the same way as the dotted line, i.e., by means of the first-order approximation of Eq. (13) with $A=1.61$ and $\mu=3/2$, but, instead of using the averaged value $V_0=2.93$ (as in the dotted line), we use the actual oscillatory value of V_0 taken from the last oscillation (from maximum to maximum) shown in Fig. 3(b). The way in which the solid line runs alongside the simulation results supports the view that the log-oscillatory behavior of $S_N(t)$ mainly comes from the log-oscillatory behavior of V_0.

B. Case (ii)

We also study the effect on $S_N(t)$ of choosing other lattice sites, besides the point O, as starting sites for the random walkers. To this end we performed simulations where all the N random walkers start on a site randomly selected within
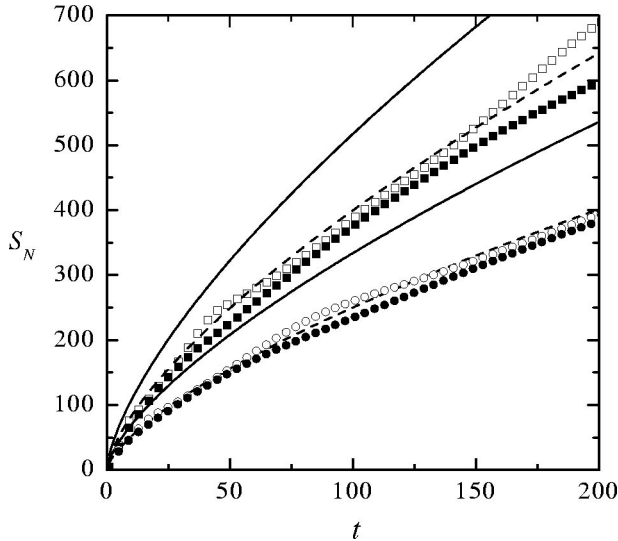
FIG. 6. The average number of distinct sites visited in the Sierpinski lattice for $N=2^8$ (filled circles) and $N=2^{13}$ (filled squares) until $t=200$ when the common origin is randomly selected from the shaded area of Fig. 2. The open symbols represent the corresponding values when the origin is the point O. The solid lines are the zeroth-order approximation and the dashed lines are the first-order approximation with $\mu=3/2$ and $A=1.61$ for, from top to bottom, $N=2^{13}$ and $N=2^8$.

FIG. 7. Dependence on N of the fractal territory $S_N(t)$ explored by N random walkers by time $t=100$ (circles) and $t=200$ (squares) when the common origin is randomly selected from the shaded area of Fig. 2. The solid lines are the zeroth-order approximation, and the dashed lines (dot-dashed lines) correspond to the first-order approximation with $\mu=3/2$ and $A=1.61$ ($\mu=1.75$ and $A=1.75$) for, from top to bottom, $t=200$ and $t=100$.

the shaded area of Fig. 2 in order to avoid the finite size effects. As expected, the fractal volume $V(r)$ and the average number of distinct sites visited $S_N(t)$ are smooth functions in this case. An estimate of V_0 by numerical fitting gives $V_0 \simeq 3.6$. The analysis of the simulation results (10^4 runs for 10^3 randomly selected starting sites) for the mean-square displacement of a single random walker is compatible with $2D \approx 0.8$ when the fit is carried out inside the time interval ($t=50, t=400$). Simulation results for $S_N(t)$ (five runs for 10^3 randomly selected starting sites) until $t=200$ for $N=1024$ are shown in Fig. 6. They are compared with the theoretical prediction of the zeroth- and first-order approximations of Eq. (13) for $\mu=3/2$ and $A=1.61$, and with the corresponding simulation results when the origin is at O [case (ii)]. Again, we find a relatively poor performance of the zeroth-order approximation, as well as substantial improvement when the first-order approximation is used, although there is still room for further enhancement. It should be noted that the performance of the two asymptotic approximations is completely analogous to that obtained for Euclidean lattices [7]. In these Euclidean media we found that the second-order asymptotic approximation gives rise to a significant improvement in the estimate of $S_N(t)$ even for relatively small values of N. It thus seems natural to conjecture that the same will occur for the Sierpinski gasket, although definitive confirmation of this guess must wait until reliable values for A, μ, and h_1 are calculated.

Finally, in Fig. 7 we show the dependence on N of $S_N(t)$ for case (ii). We have plotted two first-order asymptotic curves: for the first curve we take the usual values $\mu=3/2$ and $A=1.61$, and the new values $\mu=1.75$ and $A=1.75$ are used for the second curve. Again, one sees the great impor-
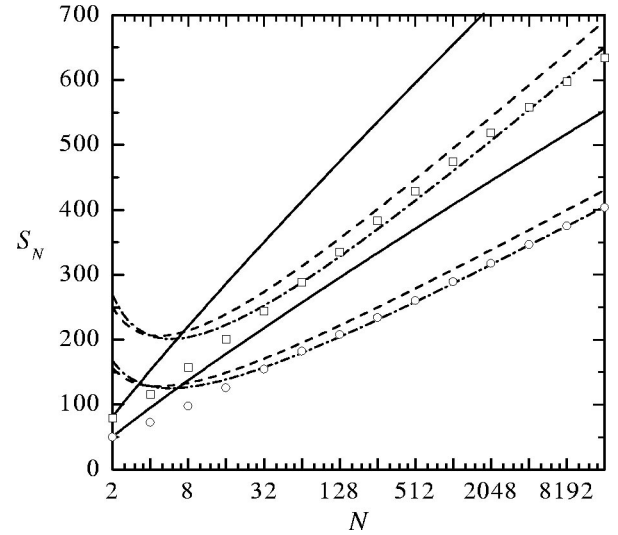
tance of the asymptotic corrective terms as they substantially improve the zeroth-order (main term) asymptotic prediction. We have used the new pair of parameters simply as another example to illustrate the gross effect of the subdominant factors of the survival probability $\Gamma_t(r)$ on the theoretical prediction of $S_N(t)$. The excellent agreement reached with $\mu=1.75$ and $A=1.75$ should not, however, be considered as an indication that they are the correct subdominant parameters of the survival probability [21].

V. NUMERICAL RESULTS FOR TWO-DIMENSIONAL PERCOLATION AGGREGATES

We have carried out simulations for the number of distinct sites visited by N independent random walkers on a typical stochastic fractal: the percolation aggregate embedded in two dimensions. The percolation aggregate has been used to characterize many disordered systems [2,12,13]. This aggregate is constructed by filling a regular lattice with ''occupied'' sites with a certain probability p. Nearest-neighbor occupied sites are supposed to be connected and form a series of clusters. At a certain critical concentration p_c an infinite cluster appears, which is called the incipient percolation aggregate or percolation cluster at criticality. In our simulations every random walker makes a jump from a site to one of its nearest neighbors placed at one unit distance in each unit time. The incipient percolation aggregates were constructed by the standard Leath method [12,22] on a square lattice with side 400 using the value $p=p_c = 0.592\,746\,0$ corresponding to site percolation in the square lattice [12]. Our simulations were carried out over 2000 aggregates generated in this way.
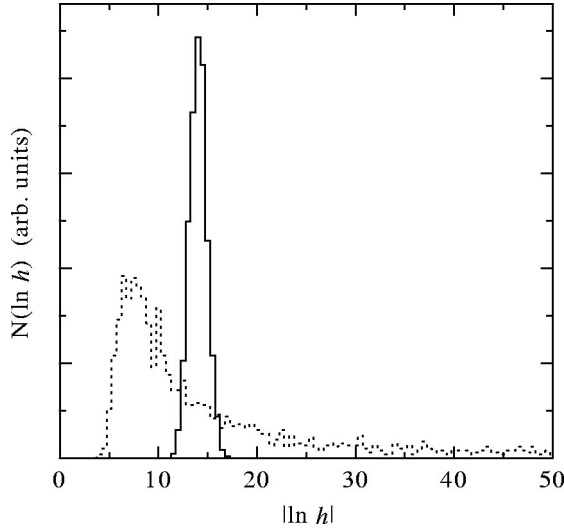
FIG. 8. Plot of the histogram $\mathcal{N}(\ln h)$ versus $|\ln h|$ in the two-dimensional incipient percolation aggregate for fixed r and t (dotted line) and for fixed ℓ and t (solid line). The values are $r=30$, $l=80$, and $t=1000$.

A. Survival probability, diffusion coefficients, and fractal volume

In order to compare the simulation results for $S_N(t)$ with the predictions of our theoretical approach, Eq. (13), we must check that the survival probability or, equivalently, the mortality function, $h(\ell,t)=1-\Gamma_t(\ell)$, really behaves in the form conjectured in Eq. (10). Moreover, we must confirm first that, for a given chemical distance ℓ, the distribution of $h(\ell,t)$ over different realizations of the incipient percolation cluster is narrow because our theoretical analysis [cf. Eq. (8)] relies on this assumption [cf. Eq. (7)]. The numerical evaluation of this quantity as well as the propagator $P(\mathbf{i},t)$ (i.e., the probability of finding a single random walker at site $\mathbf{i}$ at time t) is performed by the Chapman-Kolmogorov method (also called the exact enumeration method [2,12]). The trap is simulated by a special site belonging to the cluster that absorbs all the probability density that enters it without giving back any probability to its neighbors. In the simulation of the mortality function, we located a trap at a chemical distance $\ell=30$ in each of the 2000 percolating clusters. We repeated the experiment for traps located at a fixed Euclidean distance, $r=80$. The resulting histogram for $t=1000$ is shown in Fig. 8 (to be compared with the histogram of the propagator shown in Fig. 4 of Ref. [15]). One observes that the distribution corresponding to fixed ℓ is very narrow whereas the Euclidean version is broad and exhibits a long tail.

Figure 9 shows the chemical mean-square displacement $L^2 \equiv \langle \ell^2 \rangle = \langle \Sigma_{\text{sites}} \ell^2 P(\ell,t) \rangle$ as a function of time. The propagator in the chemical space $P(\ell,t)$ is obtained by summing $P(\mathbf{i},t)$ over all cluster sites $\mathbf{i}$ on the chemical shell situated at distance ℓ from the origin. The result is compatible with the Einstein relation Eq. (11) with $2D_\ell=1.20 \pm 0.1$ and $d_w^\ell=2.40\pm0.05$. This value for d_w^ℓ coincides with that obtained in Ref. [15] and is in agreement with the value reported in Refs. [2,23].
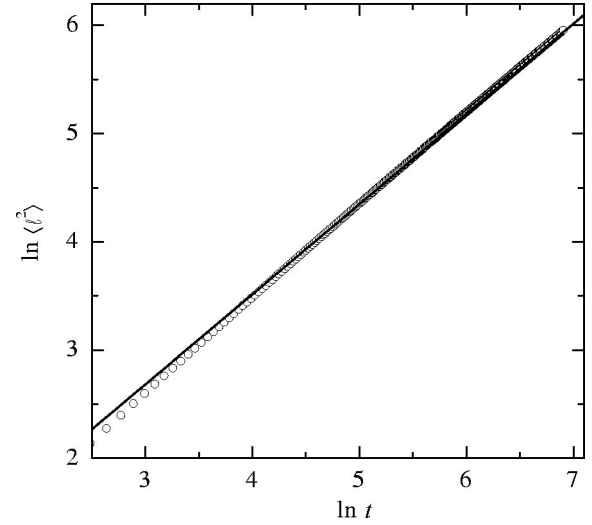

FIG. 9. Plot of $\ln\langle \ell^2 \rangle$ versus $\ln t$, $\langle \ell^2 \rangle \equiv L^2$ being the chemical mean-square displacement of a single random walker calculated over 2000 two-dimensional percolation clusters at criticality. The line represents the function $L^2=2D_\ell t^{2/d_w^\ell}$ with $2D_\ell=1.2$ and $d_w^\ell=2.4$.

In Fig. 10 we plot $\ln[-\ln h(\ell,t)]$ versus $\hat{\xi} \equiv \ell/t^{1/d_w^\ell}$ with $\ell=80$ and, according to the previous discussion, $d_w^\ell=2.40$. If the conjecture in Eq. (10) is right, we can take $h(\ell,t) \sim \exp(-\hat{c}\hat{\xi}^v)$ as a first approximation, and hence should observe the linear behavior $\ln[-\ln h(\ell,t)] \sim \ln\hat{c}+v\hat{\xi}$ with $\hat{\xi} = \sqrt{2D_\ell}\xi$ and $\hat{c}=c/(2D_\ell)^{v/2}$. Certainly the plot seems linear except for a portion in the range $\hat{\xi} \gtrsim 2.2$. This is a finite size effect (already analyzed in the case of the two-dimensional Sierpinski gasket in Ref. [17]) associated with
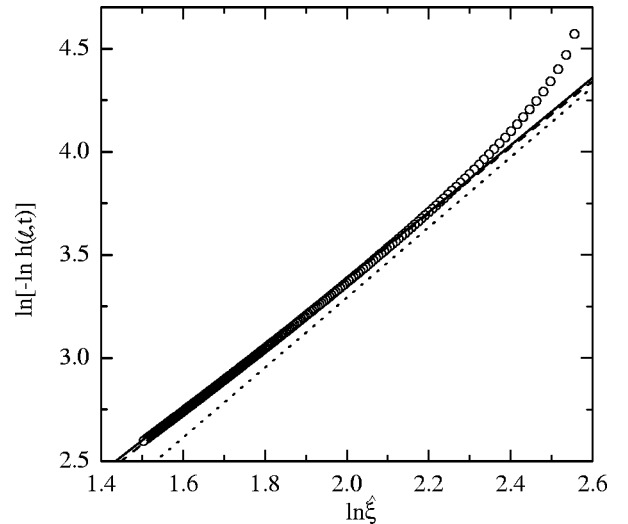

FIG. 10. Plot of $\ln[-\ln h(\ell,t)]$ versus $\ln \hat{\xi}$ averaged over 2000 two-dimensional incipient percolation clusters. The trap was always placed at a site at distance $\ell=80$ from the origin. The lines represent the functions $h(\ell,t)=\hat{A}\hat{\xi}^{-\mu v}\exp(-\hat{c}\hat{\xi}^v)$ with $\hat{A}=1.1, \mu=0.8, \hat{c}=0.9, v=1.7$ (solid line), $\hat{A}=1, \mu=0, \hat{c}=1.2, v=1.6$ (dashed line), and $\hat{A}=1, \mu=0, \hat{c}=0.9, v=1.7$ (dotted line).

the existence of a minimum arrival time corresponding to a random walker who travels ''ballistically'' along a chemical path from the origin to the trap, which in turn implies a maximum available value of $\hat{\xi}$ in the simulations (in our simulations this maximum value is $80/80^{1/d_w^\ell} \simeq 12.9$). A reliable interval for numerical fits should exclude this very short time regime. A linear fit in the interval $1.6 \leqslant \ln \hat{\xi} \leqslant 2.17$, corresponding to $200 \leqslant t \leqslant 1000$, gives the values $\hat{c} = 1.2 \pm 0.1$, i.e., $c = 1.3 \pm 0.1$, and $v = 1.6 \pm 0.05$. The dashed line in Fig. 10 corresponds to these values. The good agreement with numerical values in the above interval seems to assure the correctness of the approximation $h(\ell, t) \sim h_a(\ell, t) = \exp(-\hat{c} \hat{\xi}^v)$ with the values of c and v given above. However, the solid line in Fig. 10 is a challenge to this interpretation: one sees that the function $h_b(\ell, t) = \hat{A} \hat{\xi}^{-\mu v} \exp(-\hat{c} \hat{\xi}^v)$ with $v = 1.70$, $\hat{c} = 0.9$ (i.e., $c = 1.05$), $\mu = 0.8$, and $\hat{A} = 1.1$ is as good as $h_a(\ell, t)$. Indeed, $h_b(\ell, t)$ is more consistent from a theoretical point of view than $h_a(\ell, t)$ because the expected theoretical value of v corresponding to $d_w^\ell = 2.40$ is $v = d_w^\ell/(d_w^\ell - 1) = 1.71$, which is in better agreement with the exponent $v = 1.7$ of $h_b(\ell, t)$ than with the exponent $v = 1.6$ of $h_a(\ell, t)$. Finally, it should be noticed that the values $c = 1.05$, $v = 1.70$ are also in agreement with the corresponding parameter values of the propagator [15], thus supporting the guess made in Sec. III [see below Eq. (11)] that the dominant exponential term of the propagator and of the mortality function are the same. This leads us to consider that the set of parameters $\hat{c} = 0.9, v = 1.7, \mu = 0.8$ is more reliable than $\hat{c} = 1.2, v = 1.6, \mu = 0$. Obviously, further intensive (and extremely time consuming) computer simulations for the mortality function would be required in order to reliably determine the values of the parameters that appear in Eq. (10) and in the asymptotic corrections of $S_N(t)$.

We have also evaluated numerically the fractal volume in terms of the chemical distance $V(\ell)$, i.e., the number of lattice sites inside a circumference (in chemical space) of radius ℓ. The results are shown in Fig. 11. A good fit to $dV(\ell) = d_\ell V_0 \ell^{d_\ell - 1} d\ell$ is found with $V_0 = 1.1 \pm 0.2$ and $d_\ell = 1.65 \pm 0.05$. Taking into account that $d_f = 91/48$, we deduce that $d_{\min} = d_f/d_\ell = 1.15 \pm 0.05$, which agrees with previous estimates [2,12].

B. Simulation results: Territory covered by N random walkers on the percolation aggregate

In our simulations we calculated $S_N(t)$ by averaging over 100 runs per cluster over 2000 percolation clusters. The maximum time considered was $t = 1000$.

According to Eqs. (13) and (14), the quotient $S_N(t)/(\ln N)^\gamma$ with $\gamma = d_\ell/v$ is only a function of t. In Fig. 12 the logarithm of that quotient is plotted versus $\ln t$ for several values of N. The data collapse and the slope close to 0.66 seems to support Eq. (1) with $\gamma = d_\ell/v = 0.97$, which is in agreement with similar recent results for the three-dimensional percolation aggregate [8]. The collapse is, however, slightly poorer when the exponent $\gamma = d_f/u = 1.24$ ($d_f = 91/48$ and $d_w = 2.87$ [2,12]) proposed by Havlin *et al.* [4]
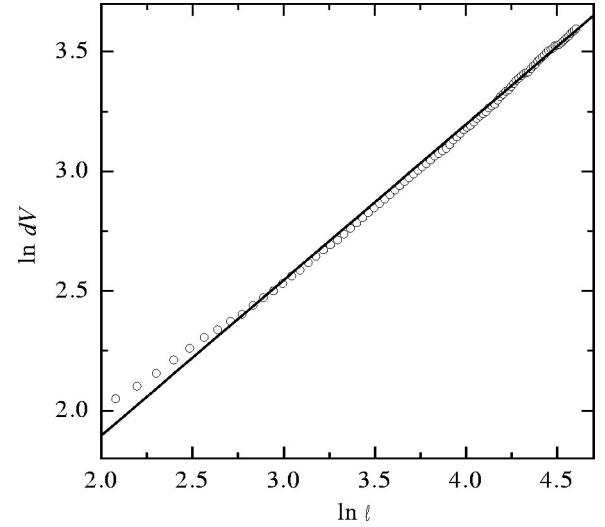
FIG. 11. Plot of $\ln dV(\ell)$ versus $\ln \ell$ in the two-dimensional incipient percolation aggregate with $dV(\ell) = V(\ell+1) - V(\ell)$ being the average number of sites in the chemical shell at distance ℓ. The line represents the function $V_0 d_\ell \ell^{d_\ell - 1}$ with $V_0 = 1.1$ and $d_\ell = 1.65$.

[see Eq. (5)] is used, as Fig. 12 shows. So, one might be led to the conclusion that the correct value of γ as defined above is d_ℓ/v. But in this analysis there was no consideration of the relatively large logarithmic corrections predicted by the asymptotic analysis presented in Sec. III, so that the reliability of the above conclusion is seriously affected by this omission.

To illustrate this point, let us now carry out the same kind of analysis with the simulation results of $S_N(t)$ when the substrate is a three-dimensional Euclidean lattice. For this case it is well known [7] that $S_N(t)$ is given by an asymptotic
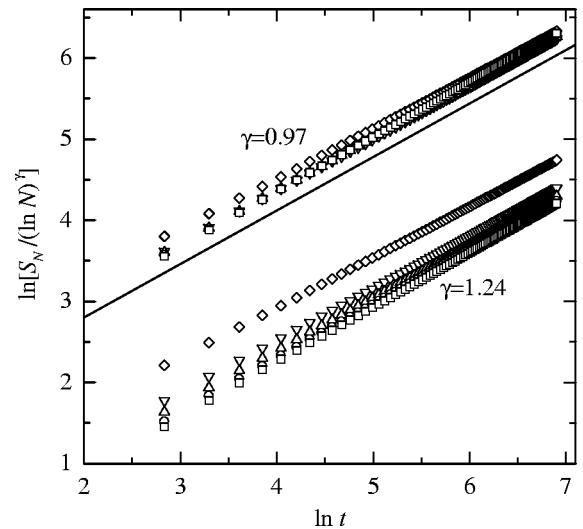
FIG. 12. Plot of $\ln[S_N(t)/(\ln N)^\gamma]$ versus $\ln t$ for $N = 2^2$ (diamond), 2^5 (down triangle), 2^8 (up triangle), 2^{11} (circle), and 2^{13} (square) in the two-dimesional percolation aggregate with $\gamma = d_\ell/v \simeq 0.97$ and $\gamma = d_f/u \simeq 1.24$. The values corresponding to $\gamma = d_\ell/v \simeq 0.97$ have been shifted up by 3/2. The line has a slope equal to $d_s/2 \simeq 0.66$.
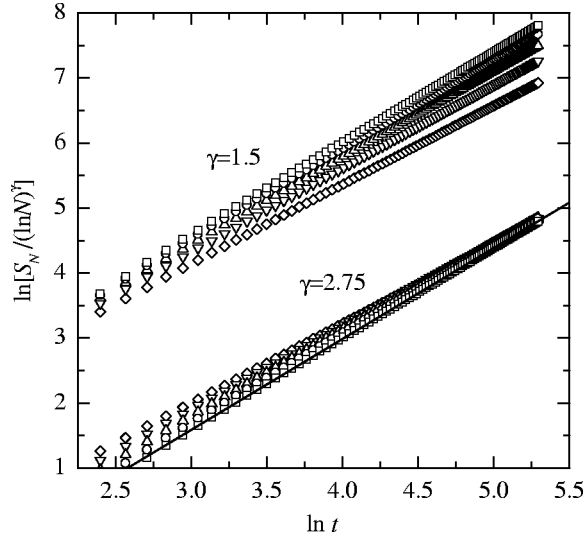
FIG. 13. Plot of $\ln[S_N(t)/(\ln N)^\gamma]$ versus $\ln t$ with $N=2^8$ (diamond), 2^{10} (down triangle), 2^{12} (up triangle), 2^{14} (circle), 2^{16} (square) for the three-dimensional Euclidean lattice with $\gamma=3/2$, and $\gamma=2.75$. The line has a slope equal to 1.4.

FIG. 14. Plot of $S_N(t)/t^{d_\ell/d_w^\ell}$ versus $\ln N$ in the two-dimensional incipient percolation aggregate for $N=2^0,2^1,\ldots,2^{13}$. The circles (squares) are the simulation results for $t=1000$ ($t=500$) averaging over 2000 aggregate realizations. The dashed (dotted) line is the zeroth-order theoretical prediction with $c=1.05$ and $v=1.7$ ($c=1.3$ and $v=1.60$) and the solid line is the first-order approximation with $c=1.05$, $v=1.7$, $\mu=0.8$, and $A=1$.

expression with the form of Eq. (13) in which the logarithmic corrective terms are very important even for very large values of N. Indeed, the main asymptotic term leads to very poor predictions for $S_N(t)$, whereas the second-order approximation ($n=2$) gives excellent agreement with numerical simulation results. The exponent γ of the main logarithmic term in N and the time exponent $d_\ell/d_w^\ell=d_s/2$ are equal to 3/2. We have plotted in Fig. 13 the quotient $S_N(t)/\ln^\gamma N$ versus $\ln t$ for several values of N taking into account that the rigorous value of γ is 3/2. We see that the collapse is far from being perfect because the logarithmic corrections have been ignored. Nevertheless, an effective (but incorrect) value of $\gamma=2.75$ yields a much better data collapse and a slope close to the theoretical value $d_s/2=1.5$. We thus conclude that the analysis of data collapse plots based on the form of the main term of quantities such as $S_N(t)$ (which typically exhibit large corrective terms) should be performed with caution. The values of the exponents estimated in this way are untrustworthy because the existence of logarithmic corrections to the main term cannot simply be ignored. The value of $\gamma=2.75$ obtained before is then only an effective way of including all these corrective terms together but the true expression involves a main term of the form $(t\ln N)^{3/2}$ times a series similar to that given in Eq. (13). These considerations should prevent us from drawing hasty conclusions from a simple view of plots such as Figs. 12 and 13.

Finally, in Fig. 14 we show the dependence of $S_N(t)$ on N and compare simulation results with the zeroth- and first-order asymptotic prediction given by Eq. (13). When the parameter set $\hat{c}=0.9, v=1.7, \mu=0.8, A=1$ (see Sec. V A) is used, we get results with a very familiar aspect as they are quite similar (although, perhaps the first-order approximation is too good) to that already found for Euclidean [7] and Sierpinski lattices (Sec. IV). This is indeed encouraging. However, when the parameters $c=1.3$ and $v=1.6$ are used, we obtain a surprising and strikingly accurate zeroth-order ap-
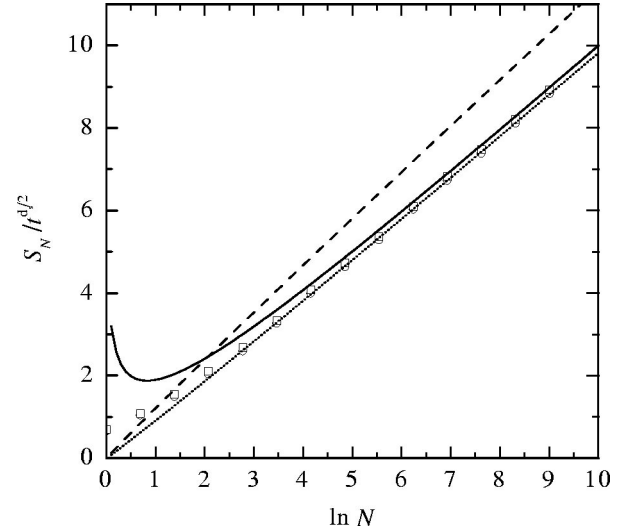
proximation. At this point, we again suspect that this last set of parameters are only effective parameters that include the influence of the true logarithmic corrective terms in the range of N simulated. Hence, Fig. 14 illustrates again, but from a different perspective, how the omission of important corrective terms could lead to finding effective parameters that, although providing excellent approximations in the (relatively short) range under consideration, are really erroneous.

VI. SUMMARY

In this paper the average fractal territory covered up to time t by N independent random walkers all starting from the same origin on fractal lattices is calculated in terms of an asymptotic series expansion, $\sum_{n=0}^{\infty}\sum_{m=0}^{n} s_{nm}(\ln N)^{d_\ell/v-n}(\ln\ln N)^m$ [see Eq. (13)], which is formally identical to those obtained for Euclidean lattices. Equation (13) is obtained by assuming that (i) the average fractal volume inside a ''hypersphere'' of chemical radius ℓ grows as $V_0\ell^{d_\ell}$, (ii) the distribution of the short-time survival probability of a single random walker in the presence of a trap is narrow, so that Eq. (7) holds, and (iii) this short-time survival probability is asymptotically given by Eq. (10). We performed numerical simulations for the Sierpinski gasket and two-dimensional percolation aggregate at criticality which support the validity of the above assumptions. The mathematical method used to derive such a result had already been successfully applied to Euclidean lattices [7] and the fractal case is a fairly straightforward generalization when the previous conditions are fulfilled.

In order to check the goodness of the approximation, we carried out numerical simulations on a standard deterministic substrate (the two-dimensional Sierpinski gasket) and on a

standard stochastic fractal (the two-dimensional percolation aggregate at criticality) obtaining reasonable agreement with the theoretical results, especially when theoretical first-order asymptotic corrective terms are considered. The performance of the theoretical expressions discussed closely resembles that attained for Euclidean lattices. However, a more definitive check of the theoretical expressions for $S_N(t)$ that include corrective terms is hindered by the uncertainty in the value of the parameters v, c, A, μ, h_1, . . . that appear in the survival probability Γ_t. The determination of its dominant and subdominant terms by numerical (or analytical) procedures is a problem for future work which will surely be beset with the technical difficulties associated with the identification of these faint terms [17].

For stochastic fractals, the use of the chemical distance turns out to be fundamental in our procedure because the distribution of the short-time survival probability in the chemical space is so narrow that we can safely replace the power N of the mean value of the survival probability by the mean value of the power N of the survival probability. This allowed us to easily translate the theoretical results previously derived for Euclidean lattices [7] to disordered media.

ACKNOWLEDGMENTS

APPENDIX

In this appendix we show that Eq. (10) is reasonable for those lattices (d-dimensional Sierpinski fractals, Given-Mandelbrot curve, one-dimensional lattice, hierarchical diamond lattice, . . .) where the renormalization procedure implemented in Refs. [19,20] can be set up. The argument is as follows. Let $\mathbf{r}_n^{(i)}$, $n=1,\dots,z$ with $r^{(i)}=|\mathbf{r}_n^{(i)}|$, be the position of the z nearest neighbors of the site at $\mathbf{r}=\mathbf{0}$ in the fractal lattice decimated i times (see Fig. 2), and let $h(t,r^{(i)})$ be the probability that, in the time interval $[0,t]$, a single diffusing particle that starts at $\mathbf{r}=\mathbf{0}$ is absorbed by *any* of the traps located at its z nearest neighbors placed at the sites $\mathbf{r}_n^{(i)}$. For large values of $\xi\sim r^{(i)}/R$, i.e., for relatively short times, the event ''the random walker arrives for the first time at site $\mathbf{r}_n^{(i)}$'' and the event ''the random walker arrives for the first time at site $\mathbf{r}_m^{(i)}$'' are (almost) independent so that $h(t,r^{(i)})$ is

(almost) the sum of the probability [given by $1-\Gamma_t(r^{(i)})$] of each of the z individual events, i.e., $h(t,r^{(i)})\simeq z[1-\Gamma_t(r^{(i)})]$. Of course, the two events are not fully independent because the random walker could first arrive at site $\mathbf{r}_m^{(i)}$ after passing by the site $\mathbf{r}_n^{(i)}$. However, this is very unlikely because $\mathbf{r}_n^{(i)}$ and $\mathbf{r}_n^{(i)}$ are separated by distances of order of $r^{(i)}$ so that the fraction of random walkers that, after arriving at $\mathbf{r}_n^{(i)}$, travel to $\mathbf{r}_m^{(i)}$ in the short-time interval $[0,t]$ is of the order $\exp(-\xi^u)$, with $\xi\sim r^{(i)}/R\gg1$, because the propagator or Green's function is of this order for large values of ξ [19,17]. Therefore, it is reasonable to assume that $1-\Gamma_t(r^{(i)})=h(t,r^{(i)})/z\{1+\mathcal{O}[\exp(-\xi^u)]\}$ for large ξ. But $1-h(t,r^{(i)})$ has the form of the right-hand side of Eq. (10) with $u=d_w/(d_w-1)$, at least for the fractals that we are considering [19,20], so that Eq. (10) for $\mathbf{r}=\mathbf{r}^{(i)}$ follows. In this discussion we have assumed, in order for the renormalization analysis that leads to $\Gamma_t(r^{(i)})$ to work, that the traps were placed at those sites (such as A_g, B_g, . . . in Fig. 2) that become the nearest neighbors of the starting site after several decimations. But, in the calculation of $S_N(t)$, the function $\Gamma_t(r)$ is required for every pair of origin and destination sites in the lattice. At this point, we shall assume that the survival probability for $\mathbf{r}\neq\mathbf{r}^{(i)}$ and $\mathbf{r}=\mathbf{r}^{(i)}$ are very similar, i.e., we assume that $\Gamma_t(r^{(i)})\simeq\Gamma_\tau(r)=\Gamma_\tau(\xi)$ when $r/\tau^{1/d_w}=r^{(i)}/t^{1/d_w}=(2D)^{1/2}\xi$ for large ξ, with $\Gamma_t(\xi)$ given by Eq. (10). To the best of our knowledge this problem has not been studied and will require a specific and detailed simulation analysis that is not the object of the present work. Nevertheless, previous simulations on the Sierpinski gasket of other statistical quantities closely related to $\Gamma_t(r)$, such as the propagator (or Green's function) [17], enable us to affirm with confidence that the parameters c and u remain unchanged over the whole Sierpinski lattice whereas the subdominant ones, A, μ, h_1, . . . , do not. In this paper we use $c=0.981$ [19,17] and $u=d_w/(d_w-1)\simeq1.756$ for the two-dimensional Sierpinski gasket (values of c for other fractals can be found in Ref. [19]) but we must bear in mind that the actual values of A, μ, and h_n, $n=1,2,\dots$ will likely differ from those obtained by renormalization techniques (namely, $A=0.61$, $\mu=1/2$, $h_1=-0.56$) as these latter correspond to the special placing of the traps and origins. We must also point out that, as shown below, since the parameters h_n, $n=1,2,3,\dots$, only contribute to the second- and higher-order series terms of $S_N(t)$ and since the real value of even the first-order asymptotic term is uncertain, the values of these parameters are not considered in this paper.

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