Numerical Simulations of Random Phase Sine-Gordon Model and Renormalization Group Predictions

David J. Lancaster

Department of Computer Science
Westminster University

Juan J. Ruiz-Lorenzo

Departamento de Física
Universidad de Extremadura
Instituto de Biocomputación y
Física de los Sistemas Complejos [BIFI](UZ)

D.J.Lancaster@westminster.ac.uk, ruiz@unex.es

http://users.wmin.ac.uk/∼lancasd, http://www.unex.es/eweb/fisteor/juan

Rome, April 4 2006.

Summary

1. The model
2. Experimental realizations
3. Perturbative Renormalization Group
4. Numerical Simulations I: Static
5. Numerical Simulations II: Dynamics
6. Our Approach to the problem
7. Computation of non universal quantities: RG analysis à la Wilson.
8. Results
9. Conclusions

The Model

1) Continuum Version on the lattice

$$\mathcal{H} = \frac{\kappa}{2} \sum_{<i,j>} (\phi_i - \phi_j)^2 - \lambda \sum_i \cos 2\pi(\phi_i - \eta_i)\,.$$

ϕ_i and $\eta_i \in \mathbb{R}$. $\eta_i \in [0,1)$ uniform distributed quenched disorder.

2) Discrete Version (formally $\lambda \to \infty$ in the continuum version)

$$\mathcal{H} = \frac{\kappa}{2} \sum_{<i,j>} (h_i - h_j)^2\,.$$

$h_i = n_i + d_i$, $n_i \in \mathbb{Z}$, $d_i \in [0,1)$ uniform distributed quenched disorder.

3) Continuum Version on a $2d$ Continuum

$$\mathcal{H} = \frac{\kappa}{2} \int d^2x\,(\partial_\mu \phi)^2 - \lambda \int d^2x\, \cos 2\pi(\phi - \eta)\,.$$

$\eta(x)$ is flat distributed.

Experiments

1) Vortex lines in Type II Superconductors [Continuum Version]

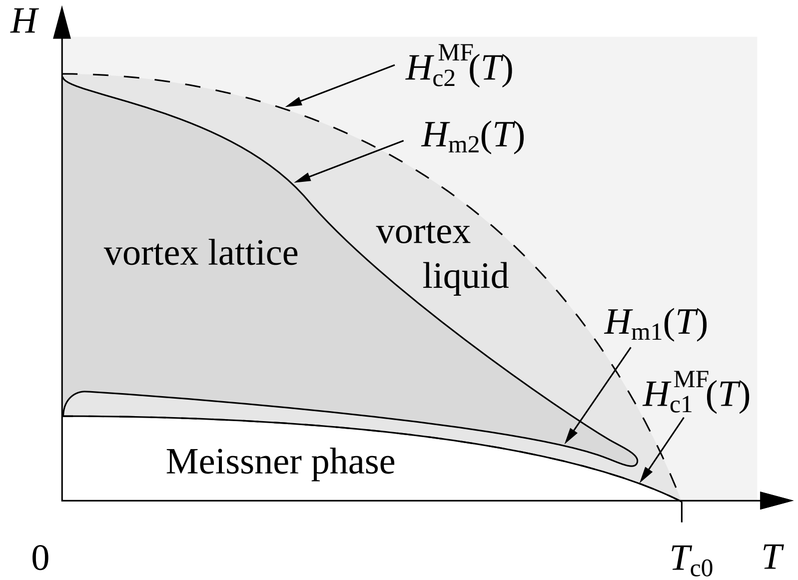

Figure 1: Phase diagram of a type II superconductor (from Nattermann et al.).

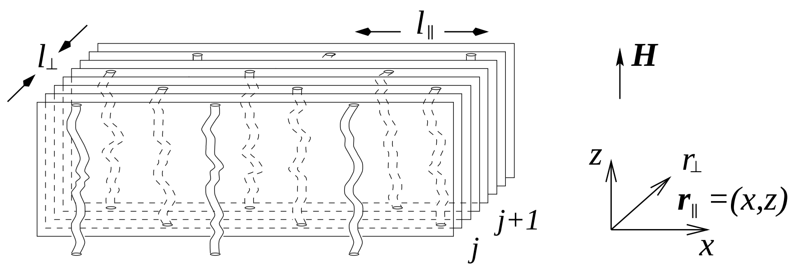

Figure 2: Flux line array in a layered superconductor in a parallel magnetic field (H) (from Kierkfeld et al.).

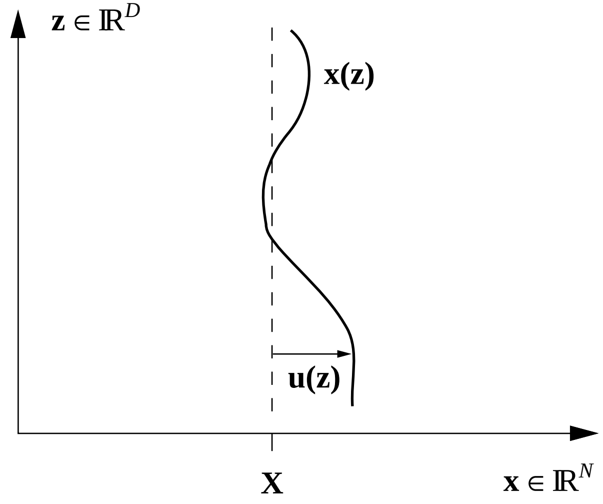

Figure 3: One flux line.

$$\mathcal{H} = \mathcal{H}_{\rm el} + \mathcal{H}_{\rm pin}\,.$$

$$\mathcal{H}_{\rm el} = \int d^2r\, \left(\frac{c_{11}}{2}(\partial_x u)^2 + \frac{c_{44}}{2}(\partial_z u)^2\right)\,.$$

$$\mathcal{H}_{\rm pin} = \int d^2r\, \rho_u(\boldsymbol{r})V(\boldsymbol{r})\,.$$

c_{11} and c_{44} are the compression and tilt elastic constants, $\rho_u(\boldsymbol{r})$ is the flux line density and $V(\boldsymbol{r})$ is the random pinning potential: $\overline{V(\boldsymbol{r})} = 0$ and $\overline{V(\boldsymbol{r})V(\boldsymbol{r'})} = \Delta(x-x')\delta(z-z')$ with $\Delta(x)$ Gaussian.

We can write the density of vortices as:

$$\rho_u(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} \delta(x - X_n - u_n(z))$$

and then by using the Poisson summation formula

$$\rho_u(\boldsymbol{r}) \simeq \frac{1}{a} \left(1 - \partial_x u(\boldsymbol{r}) + 2\cos\left(Q_m(x - u(\boldsymbol{r}))\right)\right) .$$

So, the pinning energy, for distances >> a (lattice spacing), can be written as

$$\mathcal{H}_{\text{pin}} = \int d^2r \left(-\frac{1}{2\pi} \partial_x u(\boldsymbol{r}) V(\boldsymbol{r}) + g_0 \sum_{m \geq 1} \cos\left(m u(\boldsymbol{r}) - \alpha_m(\boldsymbol{r})\right) \right) ,$$

where $\exp(i\alpha_m(\boldsymbol{r}))$ are Gaussian random phases with zero mean.

Rescaling the field, we can finally write

$$\mathcal{H} = \frac{J}{2} \int d^2r \, (\nabla\phi)^2 + \int d^2r \left\{ V(\phi(\boldsymbol{r}), \boldsymbol{r}) - \boldsymbol{\mu}(\boldsymbol{r}) \nabla\phi \right\}$$

where

$$\overline{\boldsymbol{r}} = \overline{V} = 0$$

$$\overline{\mu_i(\boldsymbol{r})\mu_j(\boldsymbol{r'})} \propto \delta_{ij} \delta_\xi(\boldsymbol{r} - \boldsymbol{r'})$$

$$\overline{V(\phi, \boldsymbol{r}) V(\phi', \boldsymbol{r'})} \propto \cos(\phi - \phi') \delta_\xi(\boldsymbol{r} - \boldsymbol{r'})$$

Figure 4: Scanning electron micrograph of a high-Q mechanical oscillator with a hexagonal single crystal of the superconductor $2H-NbSe_2$ mounted on the top. The oscillator has three layers, the top one is fixed to the substrate by two springs. The thickness of the sample is 1.5μm which is similar to the thickness of the paddle (1999) (from Bolle et al.).

2) Growth of surfaces on a disordered substrate [Discrete Version]

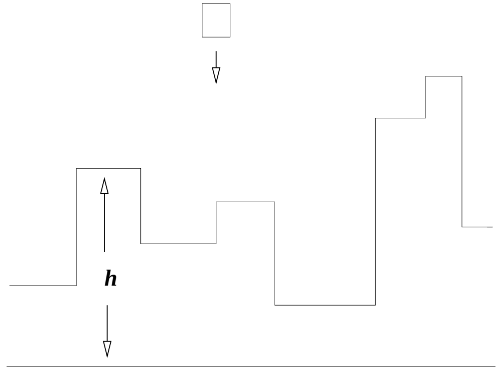

Figure 5: Growth of a surface on a plane substrate (SOS).

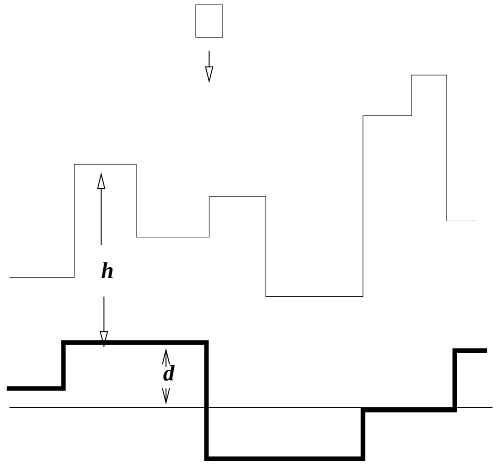

Figure 6: Growth of surfaces on a disordered substrate.

Perturbative Renormalization Group

The replicated Hamiltonian reads:

$$\beta\mathcal{H} = \int d^2x \, \frac{1}{2} \sum_{\alpha\beta} K_{\alpha\beta} \partial\phi_\alpha \partial\phi_\beta - \frac{g}{a^2} \sum_{\alpha\beta} \cos 2\pi(\phi_\alpha - \phi_\beta) \, .$$

The kinetic term is parameterized as:

$$K_{\alpha\beta} = K\delta_{\alpha\beta} + (K - \tilde{K})(1 - \delta_{\alpha\beta}) \, ,$$

with: $\tilde{K}(0) = K(0) = 2\beta$ and the bare value of g is related to the parameters of the simulated Hamiltonian by: $g(0) = (\beta\lambda/2)^2$.

The RG equations are:

$$\begin{aligned} \frac{dg}{dl} &= 2\tau g - Cg^2 \, , \\ \frac{dK}{dl} &= -Ag^2 \, , \\ \frac{d\tilde{K}}{dl} &= 0 \, . \end{aligned}$$

$\tau = (T - T_c)/T_c$. The critical temperature is at $T_c = 2/\pi$ (hence, $\tilde{K}_c = \pi$).

One can obtain these RG equations in a variety of ways; a Coulomb gas approach, conventional field theory, conformal field theory, with the help of the exact renormalization group and in our case using RG à la Wilson.

The phase diagram consists of two regions.

1) Above the critical temperature ($\tau < 0$), g flows to zero and the theory is Gaussian.

2) Below the critical temperature ($\tau > 0$), g flows to a non-trivial fixed point $g^* = 2\tau/C$. The position of this point on the line of fixed points depends on the temperature.

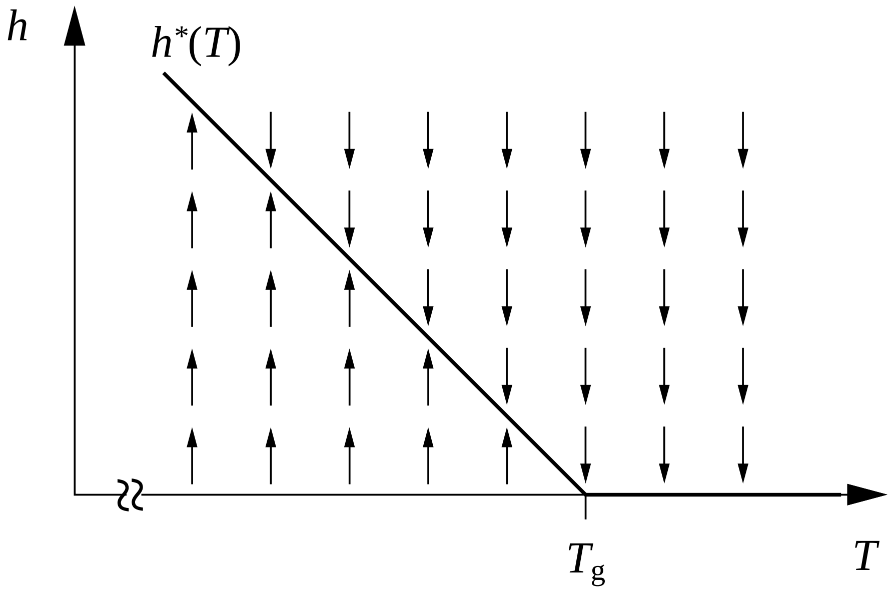

Figure 7: Renormalization group flux. Notice that $h = g$ in this plot (from Nattermann et al.).

The solution of the RG equations is:

$$g(l) = \frac{g(0)e^{2\tau l}}{1+\chi(e^{2\tau l}-1)} .$$

$$K(l) = K(0) - D\tau \left(\log\left(1+\chi(e^{2\tau l}-1)\right) - \chi(1-\chi)\frac{e^{2\tau l}-1}{1+\chi(e^{2\tau l}-1)} \right) ,$$

where $D = 2A/C^2 = 1/T$, is universal as will be discussed in the next section. $\chi = g(0)C/2\tau$.

The model has strong finite size effects. Near the critical point ($\tau < 0$), the value of g renormalized to the lattice size scale (L) is

$$g(L) \simeq \frac{1}{L^{2|\tau|}} \simeq \frac{1}{\log L} ,$$

and

$$K(L) = \mathrm{Cte} + O(\frac{1}{\log L}) .$$

These $1/\log L$ corrections to the asymptotic values imply that the fixed point values will only be obtained on very large lattices as in the ϕ_4^4 field theory in which the renormalized constant goes to zero as $1/\log L$.

Numerical Simulations I: Static

The correlation function is defined as:

$$G(r) \equiv \overline{\langle (\phi_r - \phi_0)^2 \rangle}$$

Analysis:

$$G(r) = b_1 P_L(r) + b_2 P_L^2(r)\,.$$

b_1 and b_2 are the fit parameters and $P_L(r)$ is the Gaussian correlator on a lattice of size L,

$$P_L(r) = \frac{1}{2L^2} \sum_{n_1=1}^{L-1} \sum_{n_2=0}^{L-1} \frac{1-\cos(\frac{2\pi r n_1}{L})}{2-\cos(\frac{2\pi n_1}{L})-\cos(\frac{2\pi n_2}{L})} \simeq \frac{1}{2\pi} \log(2\sqrt{2} e^{\gamma_E} r)\,.$$

γ_E is the Euler-Mascheroni constant.

RG predictions for the correlator are:

1) $T > T_c$ $[b_1 = T,\, b_2 = 0]$.

$$\overline{< (\phi(r) - \phi(0))^2 >} \approx T \log(2\pi r/a)\,.$$

2) $T < T_c$ $[b_2 = 2\tau^2]$.

$$\overline{< (\phi(r) - \phi(0))^2 >} \approx \frac{D\tau^2}{\pi \tilde{K}^2} \log^2(2\pi r/a)\,.$$

The leading $\log^2 r$ term has universal coefficient $D(\tau/\tilde{K})^2/\pi = \tau^2/2\pi^2$ (recall that $D = 2A/C^2$).

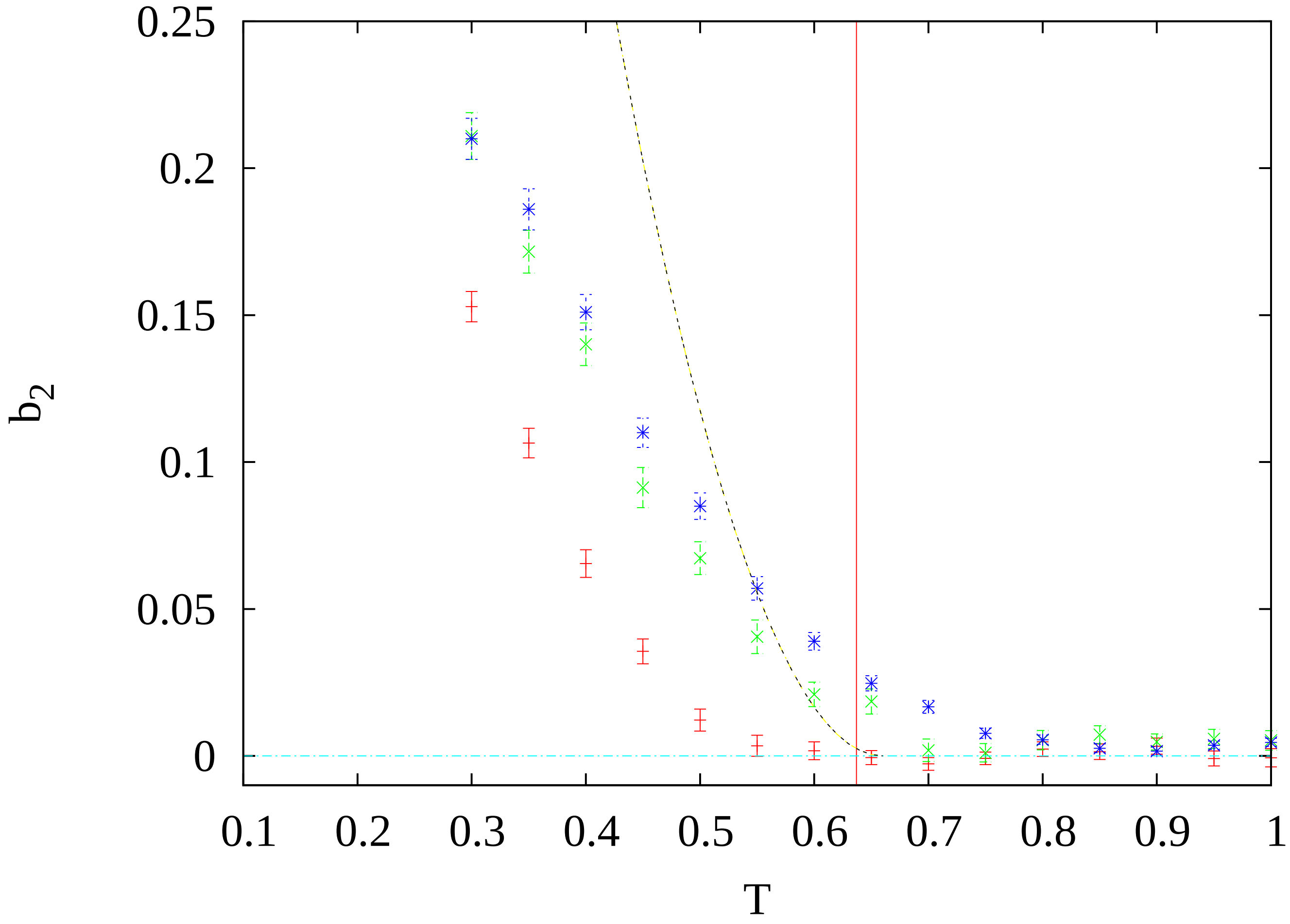

Figure 8: Super-rough component (proportional to the coefficient of $\log^2 r$) against temperature obtained in numerical simulations (using APE100).

Numerical Simulations I: Dynamics

The RG prediction for the dynamical critical exponent (z) is $z = 2$ for $T > T_c$ and in the low temperature phase:

$$z = 2 + 2e^{\gamma_E}\left(1 - \frac{T}{T_c}\right) .$$

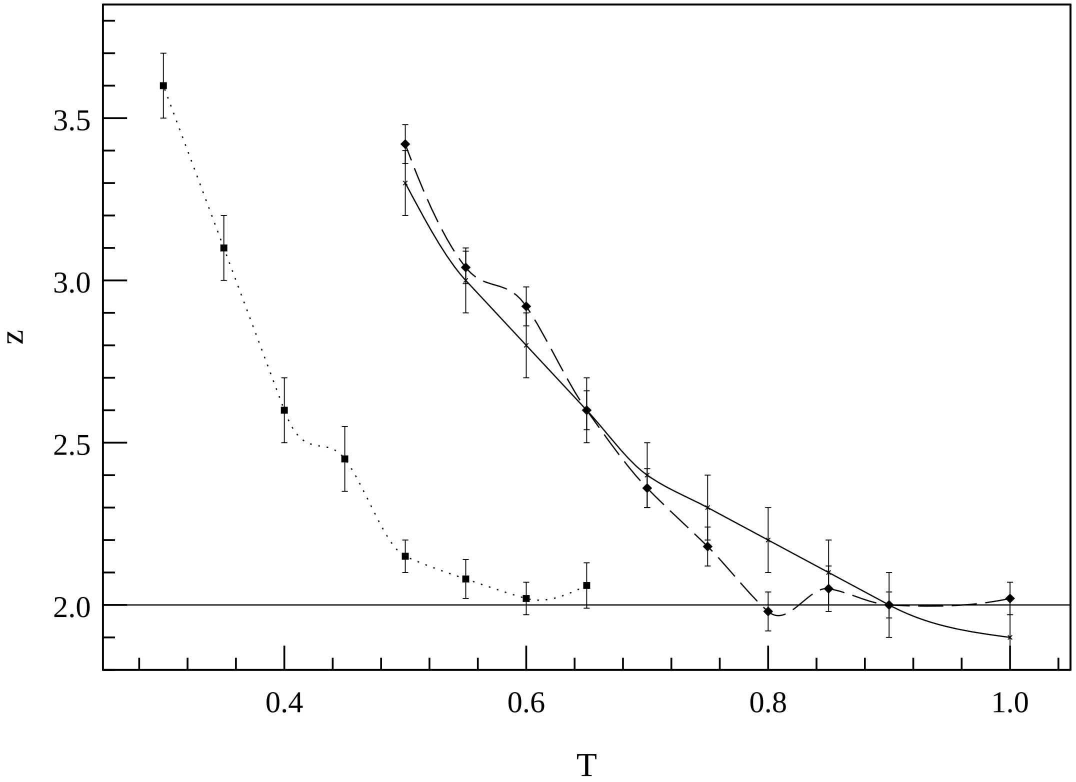

Figure 9: Dynamical critical exponent (z) as a function of temperature obtained in numerical simulations.

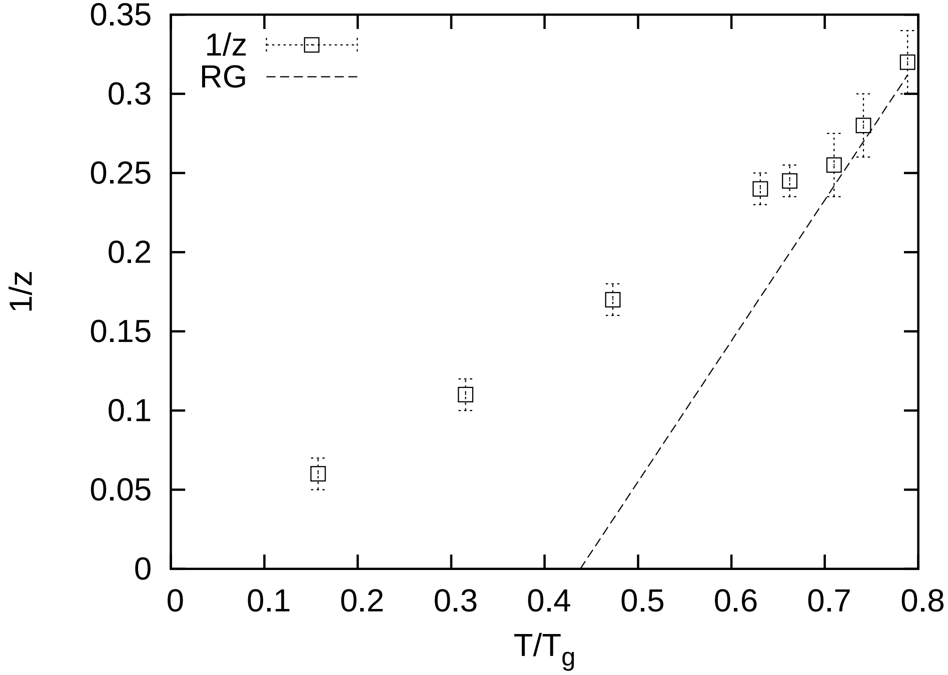

Figure 10: Dynamical critical exponent (z) as a function of temperature obtained in numerical simulations on a wide range of temperatures (from Schehr and Rieger)

Our Approach to the problem

- We study not the continuum model on the lattice, instead we study the continuum-continuum model. The Perturbation Theory on the lattice of this model is really complicated (for us!).
- The effect of removing the lattice is to induce “new” irrelevant operators.
- In the replicated framework, the average over the disorder induces a two replica term

$$\cos(2\pi(\phi_\alpha - \phi_\beta)) \, ,$$

but also higher orders. We neglect these higher order since they are irrelevant operators.

- Hence, our stating point is

$$\beta\mathcal{H} = \int d^2x \left\{ \frac{1}{2} \sum_{\alpha\beta} K_{\alpha\beta} \partial\phi_\alpha \partial\phi_\beta - \frac{g}{a^2} \sum_{\alpha\beta} \cos 2\pi(\phi_\alpha - \phi_\beta) \right\} \, .$$

- We will try to keep the lattice substrate by using the lattice propagator (as possible) in the RG calculation.

Irrelevant Operators (in detail)

1) Induced by the discrete nature of the squared lattice.

$$\phi(r+a^\mu)=\phi(r)+a\partial_\mu\phi+\frac{1}{2}a^2\partial_\mu^2\phi+\frac{1}{6}a^3\partial_\mu^3\phi+O(a^4)\,,$$

(not sum over repeated index, a^μ is a lattice vector in a given direction of the lattice (x or y)) which induces in the Hamiltonian the terms:

$$g_2\int d^dx\;(\partial_\mu^2\phi)^2+g_3\int d^dx\;(\partial_\mu^3\phi)^2+g_4\int d^dx\;(\partial_\mu\phi)(\partial_\mu^2\phi)+g_5\int d^dx\;(\partial_\mu\phi)(\partial_\mu^3\phi)+g_6\int d^dx\;(\partial_\mu^2\phi)(\partial_\mu^3\phi)+O(a^8)\,.$$

The β functions to the leading order are

$$\frac{dg_i}{dl}=(\dim g_i)\,g_i+...\,,$$

with $\dim g_2=-d$, $\dim g_2=-d$, $\dim \mathrm{g}_3=-d-2$, $\dim g_4=-d+1$, $\dim g_5=-d$, $\dim g_6=-d-1$.

so, the g's on a scale L behave as

$$g_i(L)\sim L^{+\dim g_i}\,.$$

2) Induced by the disorder.

In addition to the two replica term, the disorder induce additional irrelevant terms in the Hamiltonian:

$$u_1 \sum_{abcd} \int d^d x \;\; \cos 2\pi(\phi_a + \phi_b - \phi_c - \phi_d) + u_2 \sum_{abcd} \int d^d x \;\; \cos 2\pi(\phi_a - \phi_b) \cos 2\pi(\phi_c - \phi_d) \,.$$

The β-functions are,

$$\frac{du_i}{dl} = 2(2\tau - 1)u_i + \dots \,,$$

And the couplings u's on a scale L and near the critical temperature behave as

$$u_i(L) \simeq L^{-2} \,.$$

Computation of non Universal and Universal Quantities

The non-universal values of A and C themselves are necessary to compare with simulation.

The regulator natural in the Coulomb gas approach used by Cardy and Ostlund gives values $C = 4\pi$ and $A = 4\pi^3$. These values are rather high and we should properly use values from a lattice regulator. A full perturbative RG computation on the lattice becomes difficult so we have resorted to the following argument.

We have used momentum shell integration following Kogut to rederive the RG equations. This technique needs a modification first pointed out in Knops-Den Outen to take account of the correct operator product expansion.

This technique yields an expression in terms of the Gaussian propagator $<\phi_\alpha(r)\phi_\beta(0)>= K^{-1}_{\alpha\beta}G_0(r)$.

$$C = \frac{8\pi^2}{\tilde{K}}\Lambda^2 \int d^2\xi\, \Lambda \frac{dG_0(\xi)}{d\Lambda} e^{\frac{4\pi^2}{\tilde{K}}[G_0(\xi)-G_0(0)]} = 4\pi e^{\Phi(\infty)} .$$

Where [Le Doussal]

$$[G_0(r) - G_0(0)] = -\frac{1}{2\pi}\log(r\Lambda) + \frac{1}{4\pi}\Phi(r\Lambda) .$$

The momentum shell integration technique requires a regulator that is sufficiently smooth to make the term $\frac{dG_0(\xi)}{d\Lambda}$ short range and thus render the expression for C well defined. The lattice propagator has this property, and we can identify $\Phi(\infty)$ using the asymptotic form of the lattice regulator:

$$[G_0(r) - G_0(0)]_{\text{lattice}} \to -\frac{1}{2\pi}\log(r\Lambda 2\sqrt{2}e^{\gamma_E}) .$$

and so,

$$C = \frac{\pi}{2}e^{-2\gamma_E} .$$

In general, we can show in our calculation that at the critical point:

$$
\begin{aligned}
C &= 4\pi \int_0^\infty dx \frac{d}{dx} e^{\Phi(x)} = 4\pi e^{\Phi(\infty)} , \\
A &= 4\pi^3 \int_0^\infty dx \frac{d}{dx} e^{2\Phi(x)} = 4\pi^3 e^{2\Phi(\infty)} .
\end{aligned}
$$

So, the ratio $D = 2C/A^2$ is Universal (i.e. it does not depend on the regularization scheme used in the computation).

Some Technical Details

The change in action as fields (denoted as h) in the momentum shell are integrated out is evaluated perturbatively in g.

At first order the Gaussian integration yields the intuitive contractions:

$$\begin{aligned}\langle \cos 2\pi(\phi_\alpha(x) - \phi_\beta(x) + h_\alpha(x) - h_\beta(x))\rangle &= \frac{1}{2}\left(O_{\alpha\beta}(x)\langle e^{2\pi i(h_\alpha(x)-h_\beta(x))}\rangle + \text{c.c.}\right) \\ &= A^2(0)\tilde{A}^2(0)\cos 2\pi(\phi_\alpha(x) - \phi_\beta(x)) ,\end{aligned} \tag{1}$$

with

$$A_{\alpha\beta}(x) = e^{-2\pi^2 G_h(x) K^{-1}_{\alpha\beta}} .$$

The second order term is not much harder. In this and all subsequent computations we do not worry about operator ordering or normal ordering and always follow the consistent prescription of the path integral.

$$\begin{aligned}\langle \cos 2\pi\left(\phi_\alpha(x) - \phi_\beta(x) + h_\alpha(x) - h_\beta(x)\right) \quad & \cos 2\pi\left(\phi_\gamma(y) - \phi_\delta(y) + h_\gamma(y) - h_\delta(y)\right)\rangle_{\text{connected}} \\ = \; & \frac{1}{2}A^4(0)\tilde{A}^{-4}(0)\left(b_{\alpha\beta\gamma\delta}(\xi)O_{\alpha\beta}(x)O_{\gamma\delta}(y) + \text{c.c.}\right) .\end{aligned} \tag{2}$$

The cosine operators are at different spatial points x and y and we have written the difference: $\xi = x - y$.

The coefficients $b_{\alpha\beta\gamma\delta}(\xi)$ take account of the connected form of the expectation:

$$b_{\alpha\beta\gamma\delta}(x) = A^2_{\alpha\gamma}(x)A^2_{\beta\delta}(x)A^{-2}_{\alpha\delta}(x)A^{-2}_{\beta\gamma}(x) - 1 .$$

For a smooth cutoff, we expect the $b(\xi)$'s to be short range and allow us to use an operator product expansion (OPE) for $O_{\alpha\beta}(x)O_{\gamma\delta}(y)$.

The relevant terms in the OPE are:

$$O_{\alpha\beta}(x)O_{\gamma\delta}(y) \sim a_1(\xi)\delta_{\alpha\delta}\delta_{\beta\gamma}\left(\partial\phi_\alpha - \partial\phi_\beta\right)^2 + a_2(\xi)\left[\delta_{\alpha\delta}(1-\delta_{\beta\gamma})O_{\beta\gamma} + \delta_{\beta\gamma}(1-\delta_{\alpha\delta})O_{\alpha\delta}\right] .$$

We compute the coefficients a_1 and a_2 consistently by using exactly the same Gaussian integration techniques we have employed throughout the computation.

$$\begin{aligned} a_1(\xi) &= -\pi^2 B^4(0)\tilde{B}^{-4}(0)\xi^2 B^{-4}(\xi)\tilde{B}^4(\xi) , \\ a_2(\xi) &= B^2(0)\tilde{B}^{-2}(0)B^{-2}(\xi)\tilde{B}^2(\xi) . \end{aligned}$$

with

$$B_{\alpha\beta}(x) = e^{-2\pi^2 G(x) K^{-1}_{\alpha\beta}} .$$

Results

General RG arguments allow us to write an equation for the correlation function $C_q(r)$ of these vertex operators:

$$C_q(r, \tilde{K}, K(0), g(0)) \equiv \langle \exp\left(iq(\phi(r) - \phi(0))\right\rangle = \exp\left(-\int_{1/r}^{1} \gamma_q(g(x))\frac{dx}{x}\right) C_q(1, \tilde{K}, K(\log r), g(\log r)) \,,$$

where $\gamma_q(g)$ is twice the (RG) dimension of the vertex operator $\exp(iq\phi(r))$. The leading terms in the perturbative expansion are:

$$\gamma_q = \frac{q^2}{2\pi}\frac{1}{\tilde{K}}\left(2 - \frac{K}{\tilde{K}}\right) + \gamma_2(q)g^2 \,.$$

Computation of the term γ_2, twice the anomalous dimension, in our RG scheme is very involved (it requires the OPE of three vertex operators) and it has not been determined in other renormalization schemes.

Taking a Taylor expansion of C_q in q and picking the quadratic term, we obtain

$$G(r, \tilde{K}, K(0), g(0)) = G(1, \tilde{K}, K(\log r), g = 0) + 2\int_{0}^{\log(r)} dl \frac{1}{2\pi\tilde{K}}\left(2 - \frac{K(l)}{\tilde{K}}\right) \,.$$

By computing $G(r)$ we can obtain b_2.

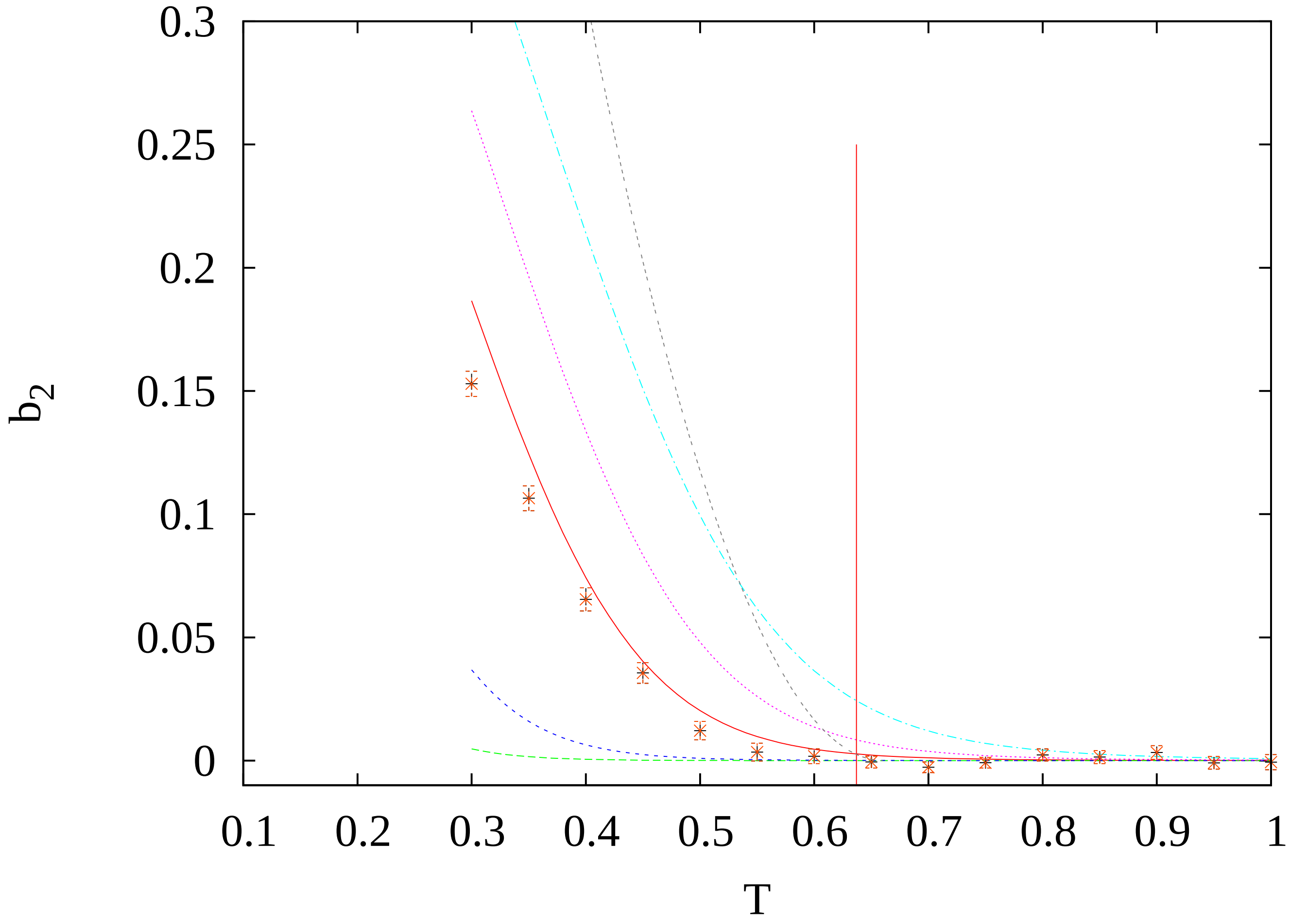

Figure 10: Our results for b_2 on a $L = 64$ lattice.

Conclusions

1. Our main result lies in the similarity between the results based on the RG prediction according to our finite size treatment and the original numerical simulation figure.
2. One can qualitatively understand the numerical observations in the framework of the RG without the need for any additional ingredient just by taking account of strong finite size effects.
3. We have obtained reasonable quantitative agreement with the numerical simulations in the regime where perturbative RG is valid, namely for small disorder strength ($\lambda = 0.5$) near the transition.
4. Above and near the critical temperature, the disorder strength renormalizes to zero and the fixed point is Gaussian. However the dependence on lattice size is extremely slow, and this behavior induces the $\log^2 r$ term above the critical temperature.
5. Open problem.Develop a finite size (time) theory for the dynamics in order to explain the dependence of z with λ.