Multifractals in Statistical Mechanics

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Outline of the Talk

- Turbulence
- Multiscaling/Multifractality
- Diluted magnets at equilibrium
- Spin glasses out-of-equilibrium
- Conclusions

Turbulence: Leonardo Da Vinci (1452-1519)

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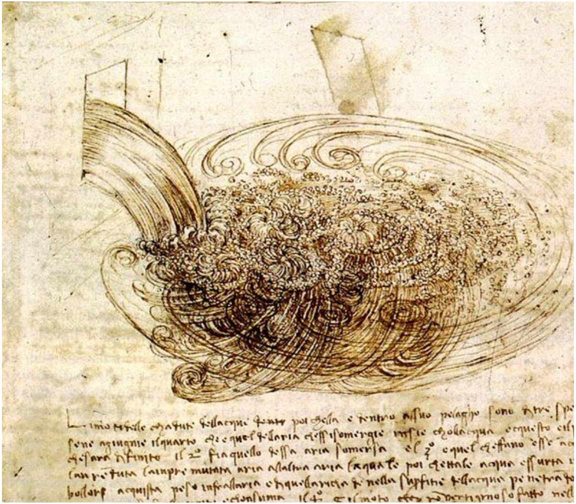
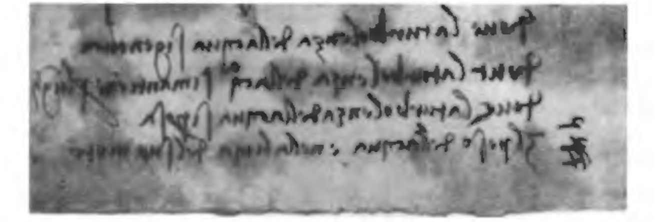
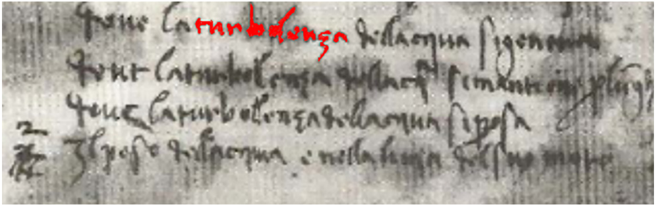


[Windsor Collection]

Turbulence: Leonardo Da Vinci (1452-1519)

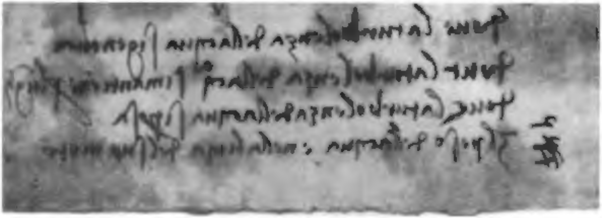


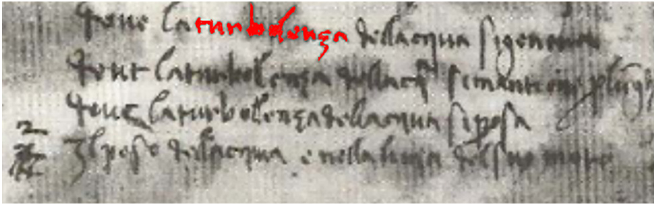


doue la turbolenza dell'acqua si genera
doue la turbolenza dell'acqua si mantiene plugho
doue la turbolenza dell'acqua si posa

[Windsor Collection]

Turbulence: Leonardo Da Vinci (1452-1519)





where the 'turbolenza' of the water is generated
where the 'turbolenza' of the water persists for a long distance
where the 'turbolenza' of the water settles

Turbulence: Navier-Stokes' equations (NSE)

- Continuity ($\rho =$const):

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

- Newton's second law:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{F}$$

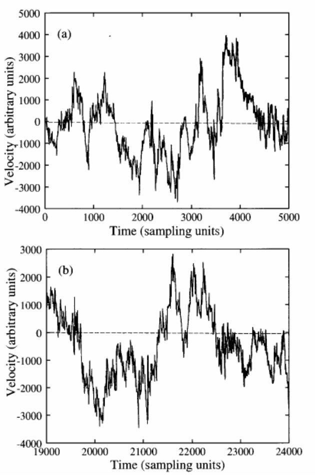
- $\boldsymbol{v}_0 = \boldsymbol{v}(t = 0) \equiv \tilde{\omega}$ plus boundary conditions.
- The (local) energy (per unit mass and time) dissipation is given by

$$\frac{dE}{dt} = -\frac{1}{2}\nu \sum_{ij} \left(\partial_i v_j + \partial_j v_i\right)^2$$

- Finally, the Reynolds number on a system of size L with typical velocity u:

$$Re = \frac{uL}{\nu}$$

Turbulence: Velocity



[U. Frisch]

Turbulence: Some properties

- Cascades for $Re > Re_c$.
- Fully developed turbulence (FDT) ($Re \gg Re_c$).
- Isotropy on local scales in FDT.
- Statistical description of the turbulence is justified (partially motivated by the study of chaos in deterministic systems).
  1. The full solution $\boldsymbol{v}(t, \boldsymbol{r}, \tilde{\omega})$ of the NSE is a stationary random function. A random function $v(t, \tilde{\omega})$ is said to be G_t-stationary (semi-group of time shifts) if for all t and $\tilde{\omega}$

$$\boldsymbol{v}(t+h, \tilde{\omega}) = \boldsymbol{v}(t, G_h \tilde{\omega})\,, \ \forall h > 0$$

  2. “Ergodic Theorem”

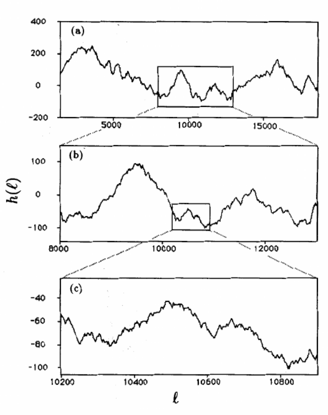
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \ v(t) = \langle v \rangle = \int dv \ \rho(v) v$$

Turbulence: Kolmogorov's theory (K41)

Two main hypothesis:

- The dissipated energy per unit mass and time does not depend on the scale.
- The turbulence is self-similar at different scales.

Self-affine curves



$$h(b\ell) = b^{\alpha} h(\ell)\,, \quad \alpha = 2 - D_F$$

[Barabási and Stanley]

Turbulence: Kolmogorov's theory (K41)

Scaling Laws:

$$\delta v_{\parallel}(\boldsymbol{r}, \boldsymbol{\ell}) \equiv \big(\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{\ell}) - \boldsymbol{v}(\boldsymbol{r})\big) \cdot \frac{\boldsymbol{\ell}}{\ell}$$

$$S_p(\boldsymbol{\ell}) \equiv \langle \big(\delta v_{\parallel}(\boldsymbol{r}, \boldsymbol{\ell})\big)^p \rangle \sim \ell^{\zeta_p}$$

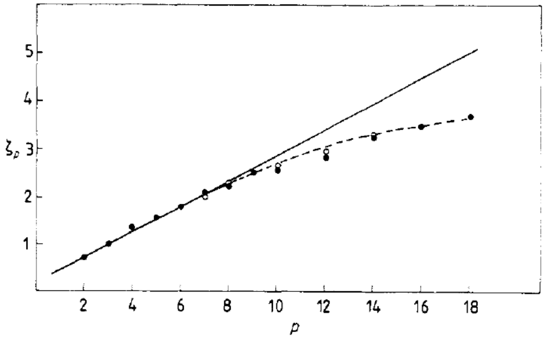
Note: Assuming homogeneity and isotropy, $S_p(\boldsymbol{\ell})$ does not depend on $\boldsymbol{r}$.

$$\zeta_p = p/3$$

In particular, Kolmogorov showed

$$S_3(\boldsymbol{\ell}) = -\frac{4}{5}\epsilon\ell$$

Turbulence: Multiscaling



[Benzi et al.]

Turbulence: Multifractals

Fully Developed Turbulence and Intermittency.

U. Frisch

CNRS, Observatoire de Nice - BP 139, 06003 *Nice Cedex, France*

Turbulence and predictability in geophysical fluid dynamics and climate dynamics, Varenna, 1983, M. Ghil, R. Benzi and G. Parisi, eds. Proceedings of the International School of Physic Enrico Fermi, North-Holland 1985

The Appendix by G. Parisi and U. Frisch *On the singularity structure of fully developed turbulence* is on pp. 84-87.

Appendix

On the singularity structure of fully developed turbulence.

with

G. Parisi

Dipartimento di Fisica, Università di Roma II « Tor Vergata » - Roma, Italia

the numerical evidence is inconclusive [26, 31]. Mandelbrot [24, 25] and others [20] have considered models with singularities concentrated on a set $\subset \boldsymbol{R}^3$ having noninteger (fractal) Hausdorff dimension. We shall here show that the data suggest the existence of a hierarchy of such sets (a « multifractal »).

Turbulence: Multifractals

- $\delta \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{\ell}) \equiv \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{\ell}) - \boldsymbol{v}(\boldsymbol{r})$ with $|\delta \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{\ell})| \sim \ell^{h(\boldsymbol{r})}$.
- The set of points $\boldsymbol{r}$ with exponent h defines a fractal set $S(h) \in \mathbb{R}^3$ with fractal dimension $D_F(h)$.
- Remember $S_p(\ell) = \langle |\delta \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{\ell})|^p \rangle \sim \ell^{\zeta_p}$
- But,

$$S_p(\ell) = \langle |\delta \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{\ell})|^p \rangle \sim \int dh\, \rho(h) \ell^{3-D_F(h)} \ell^{ph}$$

$$= \int dh\, \rho(h) e^{(\log \ell)(3-D_F(h)+ph)} \sim \ell^{\zeta_p}$$

- Since $\ell \ll 1$, a saddle point computation provides:

$$\zeta_p = \min_h (hp + 3 - D_F(h)) = h^* p + 3 - D_F(h^*)$$

with $D_F'(h^*(p)) = p$. Inverting the Legendre Transform

$$D_F(h) = \min_p (hp + 3 - \zeta_p)$$

Multifractals

- Turbulence
- Anderson localization
- Diffusion-limited aggregates
- Chaotic dynamics
- Surface growth
- Rainfall
- Human heartbeat dynamics
- Finance
- etc

Multifractals

- Turbulence
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- etc
- Spin Glasses?

3D Diluted Ising Model (DIM)

- Hamiltonian

$$\mathcal{H} = - \sum_{<\boldsymbol{x}\boldsymbol{y}>} \epsilon_{\boldsymbol{x}} \epsilon_{\boldsymbol{y}} S_{\boldsymbol{x}} S_{\boldsymbol{y}}$$

$\epsilon_{\boldsymbol{x}}$ [quenched disorder]: 1 with probability p and 0 otherwise.

- Correlation function:

$$C_q(r) = \frac{1}{pV} \sum_{\boldsymbol{x}} \overline{\langle \epsilon_{\boldsymbol{x}+\boldsymbol{r}} S_{\boldsymbol{x}+\boldsymbol{r}} \epsilon_{\boldsymbol{x}} S_{\boldsymbol{x}} \rangle^q} \sim \frac{1}{r^{\tau(q)}} \sim \Big(C_1(r) \Big)^{\zeta(q)}$$

- Notice that, as usual,

$$C_1(r) \sim \frac{1}{r^{D-2+\eta}}$$

- Then

$$\tau(q) = (D - 2 + \eta)\zeta(q)$$

- Multiscaling: $\zeta(q) \neq q$

3D Diluted Ising Model (DIM)

- Global susceptibilities:

$$\chi_q = \frac{1}{pV} \sum_{\boldsymbol{x}\boldsymbol{y}} \overline{\langle \epsilon_{\boldsymbol{x}} S_{\boldsymbol{x}} \epsilon_{\boldsymbol{y}} S_{\boldsymbol{y}} \rangle^q}$$

- Then

$$\chi_q \sim \int^L d^D x \; C_q(x) \sim L^{D-\tau(q)}$$

- Local susceptibilities:

$$\tilde{\chi}_q = \frac{1}{pV} \sum_{\boldsymbol{x}} \overline{\chi^q_{\boldsymbol{x}}}$$

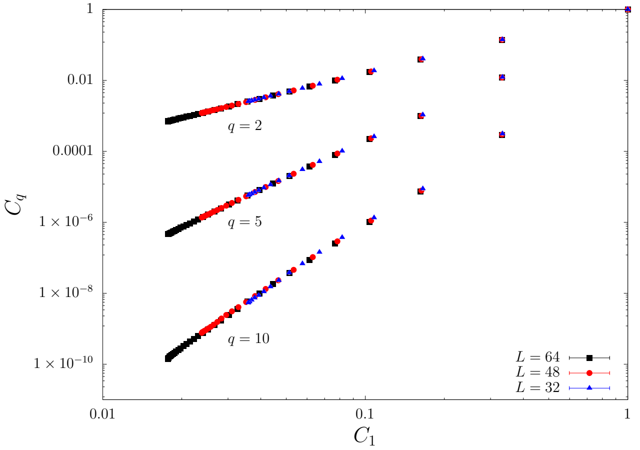
with

$$\chi_{\boldsymbol{x}} = \sum_{\boldsymbol{y}} \langle \epsilon_{\boldsymbol{x}} S_{\boldsymbol{x}} \epsilon_{\boldsymbol{y}} S_{\boldsymbol{y}} \rangle$$

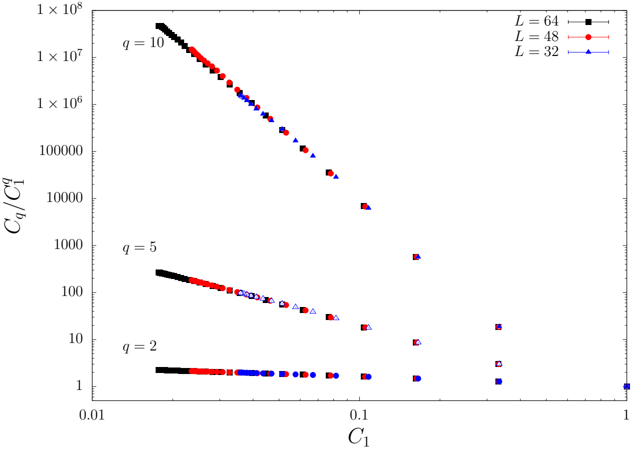
3D Diluted Ising Model: Some numerical details

- 1 sweep is composed by L single cluster Wolff updates plus a sequential full-lattice Metropolis update.
- Corrected bias for the computation of the $C_q(r)$ (using only one real replica).
- $8 \leq L \leq 128$.
- Huge number of samples.

3D Diluted Ising Model



3D Diluted Ising Model



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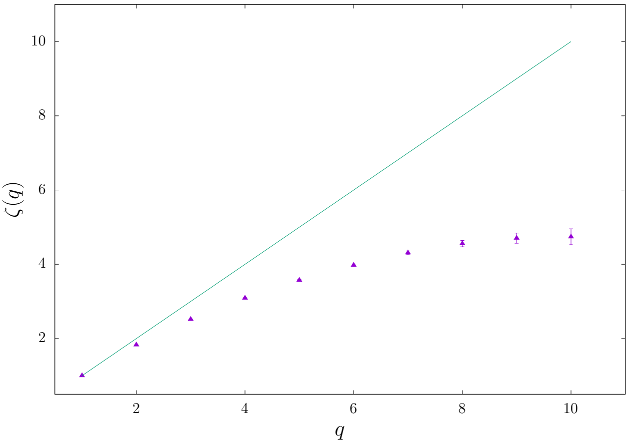


Composite Operators: Diluted Ising model (Ludwig)

- Quenched disorder (J): we need to compute $\overline{\log Z_J}$.
- Replica trick

$$\overline{\log Z_J} = \lim_{n \to 0} \frac{\overline{Z_J^n - 1}}{n}$$

- $\overline{Z_J^n}$: n copies of the system with the same disorder.
- The fields are now $\phi_a(x)$ ($a = 1, \ldots, n$).
- The Hamiltonian is

$$\mathcal{H}_{\rm eff}[\phi_a] = \int d^D x \ \left[\frac{1}{2} \sum_{a=1}^{n} (\partial_\mu \phi_a)^2 + \frac{r}{2} \sum_{a=1}^{n} \phi_a^2 + \frac{u}{4!} \left(\sum_{a=1}^{n} \phi_a^2 \right)^2 + \frac{v}{4!} \sum_{a=1}^{n} \phi_a^4 \right]$$

$$C_q(x) = \overline{\langle S_x S_0 \rangle^q} \to \langle \left[\phi_{a_1}(x) \ldots \phi_{a_q}(x) \right] \left[\phi_{a_1}(0) \ldots \phi_{a_q}(0) \right] \rangle$$

and we have generated correlations with composite operators $\to$ multiscaling (multifractality).

- It is possible to show that $\zeta(q)$ is a concave function.

Scaling of Composite Operators

- The (scaling) behavior of the operator $\phi(x)$ will be different of that of $\phi(x)^2$.
- In general, if $\phi(\lambda x) = \lambda^{-h}\phi(x)$, one gets:

$$\langle \phi(x)\phi(0) \rangle \sim \frac{1}{|x|^{2h}}$$

- but

$$\langle \phi(x)^2\phi(0)^2 \rangle \sim \frac{1}{|x|^{2h_2}}$$

  with $h_2 \neq 2h = D - 2 + \eta$.
- Physically, for example for Ising models, ϕ is the magnetization and ϕ^2 is the energy per spin and at the critical point ($D < 4$): $e \neq m^2$.
- Hence, in the DIM

$$C_q(r) \sim \frac{1}{r^{2h_q}} \quad \text{and} \quad \tau(q) = 2h_q$$

  with $h_q \neq qh$.

Davis-Cardy result

Using perturbation theory on a conformal field theory (2D), they found

$$q-\zeta(q)=\frac{q(q-1)}{2}y+O(y^2)$$

with $y=\alpha=O(\epsilon)$ (α is the specific heat exponent of the pure model and $\epsilon=D-2$).

- This result clearly shows that the DIM in $2+\epsilon$ is expected to undergo a multiscaling behavior.
- However, in order to have an accurate analytical estimate of the difference $q-\zeta(q)$ for the 3D model one would need to extend this computation to higher orders of y.

3D Edwards-Anderson Model

- Hamiltonian

$$\mathcal{H} = -\sum_{<\boldsymbol{xy}>} J_{\boldsymbol{xy}} S_{\boldsymbol{x}} S_{\boldsymbol{y}}$$

  $J_{\boldsymbol{xy}}$ quenched disorder: ± 1 with equal probability.

- Order parameter (Overlap):

$$q_{\boldsymbol{x}} = S_{\boldsymbol{x}}^{(1)} S_{\boldsymbol{x}}^{(2)} \,, \quad q = \frac{1}{V} \sum_{\boldsymbol{x}} \overline{\langle q_{\boldsymbol{x}} \rangle}$$

- Correlation function:

$$\overline{C_4^q(r = \xi(t_w), t_w)} = \overline{\langle q_{\boldsymbol{r}}(t_w) q_0(t_w) \rangle^q} \sim \frac{1}{\xi^{\tau(q)}}$$

$$\overline{C_4(\boldsymbol{r}, t_w)} \sim \frac{f(r/\xi(t_w))}{r^\theta}$$

$$C_4(r = \xi(t_w), t_w) \sim \frac{1}{\xi^\alpha}$$

- Multiscaling (α distributed via $P(\alpha) \sim \xi^{f(\alpha)}$)

$$f(\alpha) = \min_q \Big(q\alpha - \tau(q) \Big)$$

Janus II (Ferrara-Roma-UCM-Zaragoza-Extremadura, 2013)

Janus II

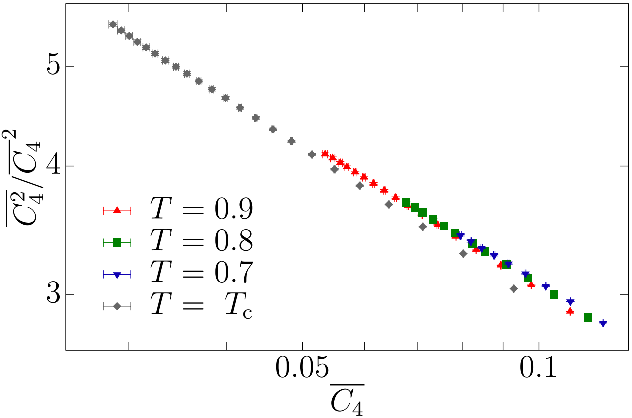
Some figures:

- Built in 2015.
- ~ 5 times most powerful than (the previous) Janus I.
- Dedicated computer optimized to simulate a wide variety of spin models.
- Flexible topology.
- 16 boards of 16 FPGA's each (one IOP and PC integrated on each board) (Virtex 7).
- Janus II has allowed us to simulate in the 1 second time region.

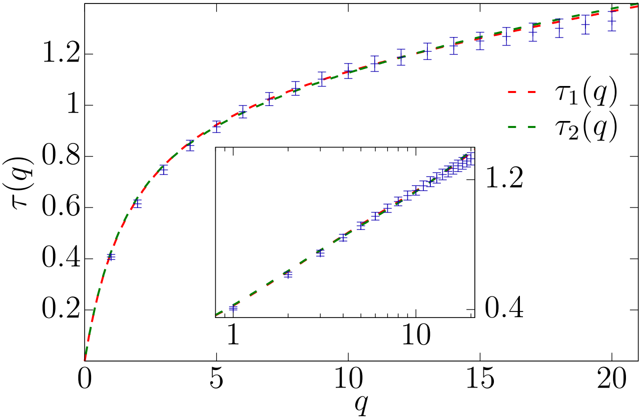
[Janus Coll, Comp. Phys. Comm. 185, 550 (2014).]

Janus Collaboration

3D Edwards-Anderson Model



3D Edwards-Anderson Model



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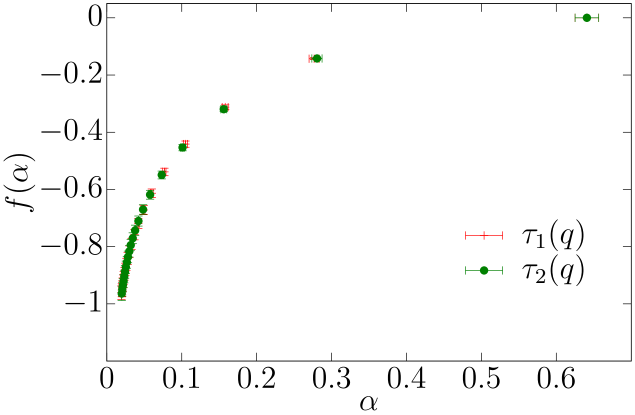
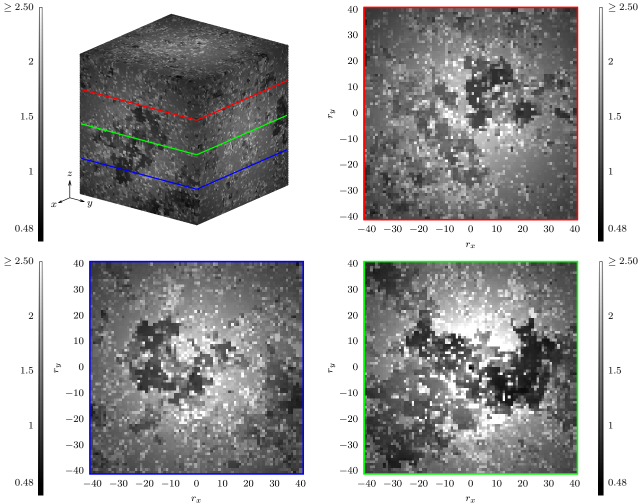


Multifractality: Spatial distribution

$$\overline{C_4(\boldsymbol{r}, t_w)} \sim \frac{f(r/\xi(t_w))}{r^{\theta}} \ , \ C_4(\boldsymbol{x}, \boldsymbol{r}+\boldsymbol{x}, t_w) \sim \frac{1}{r^{\theta M(\boldsymbol{x},\boldsymbol{r},t_w)}}$$



Conclusions

- We have characterized numerically the multiscaling properties ($\tau(q)$ or $\zeta(q)$) in the three-dimensional diluted Ising model at equilibrium and at criticallity.
- We have found mutifractal behavior on the three-dimensional Edwards-Anderson model out-of-equilibrium for $T \leq T_c$.
- Some experiments have started to look locally the spin glass phase (see for example, L. Niggli et al., Dynamic heterogeneity in the self-induced spin glass state of elemental neodymium, arXiv:2412.15916 (2024).).
- The multifractal behavior of the three-dimensional Edwards-Anderson model in equilibrium is still an open problem but likely it will show multifractality (equivalence dynamics-static).
- In order to characterize the multiscaling/multifractal properties it is compulsory to work with local operators.

(some) References

- L. D. Landau and E. M. Lifshitz, “Fluid Mechanics” (Butterworth-Heinemann, 1987).
- U. Frisch, “Turbulence: The legacy of A. N. Kolmogorov” (Cambridge University Press, 1995).
- A. Colagrossi, S. Marrone, P. Colagrossi and D. Le Touzé, Physics of Fluids 33, 115122 (2021).
- H.E. Stanley and P. Meakin, Nature 333, 405 (1988).
- A. W. W. Ludwig, Nuclear Physics B 330, 639 (1990).
- E. Marinari, V. Martin-Mayor, G. Parisi, F. Ricci-Tersenghi and J.J. Ruiz-Lorenzo. Journal of Statistical Mechanics: Theory and Experiment 013301 (2024).
- T. Davis and J. Cardy, Nuclear Physics B 570, 713 (2000).
- M. Baitsy-Jesi et al. Proceedings of the National Academy of Sciences of USA, 121(2) e2312880120 (2024).