Recombination of the Hydrogen. The decoupling of photons. (Gravitation and Cosmology)

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Badajoz, May/4/2023

The Chemical Potential

- Maxwell-Boltzmann statistics:

$$n_i = g_i e^{\beta\mu} e^{-\beta\epsilon_i} \, ,$$

where g_i is the degeneracy of the state i with energy ϵ_i and μ is the chemical potential.

- We fix the chemical potential using the condition

$$N = \sum_i n_i \to \int \frac{d^3x \, d^3p}{h^3} g_i e^{\beta\mu} e^{-\beta \frac{\boldsymbol{p}^2}{2m}} \, ,$$

where we have assumed that the ideal gas is composed by non-relativistic particles.

- Accordingly

$$n = \frac{N}{V} = \frac{g_i e^{\beta\mu}}{h^3} \int d^3p \; e^{-\beta \frac{\boldsymbol{p}^2}{2m}} \, ,$$

- Notice

$$\int d^3p \; e^{-\beta \frac{\boldsymbol{p}^2}{2m}} = \int dp_x dp_y dp_z \; e^{-\beta \frac{p_x^2}{2m}} e^{-\beta \frac{p_y^2}{2m}} e^{-\beta \frac{p_z^2}{2m}} \, .$$

The Chemical Potential

- Using
$$\int_{-\infty}^{\infty} \mathrm{d}u \, \mathrm{e}^{-\alpha u^2} = \sqrt{\frac{\pi}{\alpha}} \, ,$$
we can write
$$\int_{-\infty}^{\infty} \mathrm{d}p_x \, \mathrm{e}^{-\beta p_x^2/(2m)} = \left(\frac{2\pi m}{\beta}\right)^{1/2} \, .$$
- Hence
$$n = g_i \mathrm{e}^{\beta\mu} \frac{(2\pi m k_{\mathrm{B}} T)^{3/2}}{h^3} \, .$$
[Remember $\beta \equiv 1/(k_{\mathrm{B}} T)$.]
- And the chemical potential is given by
$$\boxed{\mathrm{e}^{\beta\mu} = \frac{n h^3}{g_i \, (2\pi m k_{\mathrm{B}} T)^{3/2}}}$$

Recombination of Hydrogen: Saha equation

- The right approach is the use of the Boltzmann equation (out of equilibrium situation).
- Recombination: $e^- + p \leftrightarrow H + \gamma$.
- We assume that

$$\Gamma(e^- + p \leftrightarrow H + \gamma; t) \gg H(t)\,.$$

  [Notice: $\Gamma = \langle n\sigma v\rangle$.]
- Ingredients: protons, electrons (relics) and photons.
- We consider e^-, p and H non relativistic ideal gases ($T \ll 0.5$ MeV).
- The chemical potentials satisfy

$$\mu_{e^-} + \mu_p = \mu_H + \mu_\gamma = \mu_H$$

- with

$$e^{\beta\mu_p} = \frac{n_p h^3}{2\left(2\pi m_p k_{\mathrm{B}} T\right)^{3/2}},$$

$$e^{\beta\mu_e} = \frac{n_e h^3}{2\left(2\pi m_e k_{\mathrm{B}} T\right)^{3/2}},$$

$$e^{\beta\overline{\mu}_H} = \frac{n_H h^3}{4\left(2\pi m_H k_{\mathrm{B}} T\right)^{3/2}},$$

where we have used $g_p = 2$, $g_{e^-} = 2$ and $g_4 = 2 \times 2 = 4$.

- All three chemical potentials (μ_p, μ_e and $\overline{\mu}_H$) are measured from the minimum of the Hamiltonian: zero for protons and electrons and in the case of the Hydrogen from its ground state, with energy -13.6 eV. So,

$$\overline{\mu}_H = E_d + \mu_H \text{ , with } E_d \equiv +13.6 \text{ eV}$$

and

$$e^{\beta\mu_H} = \frac{n_H h^3}{4\left(2\pi m_H k_{\mathrm{B}} T\right)^{3/2}} e^{-\beta E_d}.$$

- In this way, all three chemical potentials (μ_p, μ_e and μ_H) are computed from the $E = 0$ level.

- Using $\mu_p + \mu_e = \mu_H$ we get

$$\frac{n_p n_e h^3}{(2\pi k_{\rm B} T)^{3/2} (m_e m_p)^{3/2}} = \frac{n_H \mathrm{e}^{-\beta E_d}}{m_H^{3/2}} \, .$$

- At this point, we assume:
  1. $m_H \simeq m_p$.
  2. Neutral universe: $n_e = n_p$.
- Therefore

$$\frac{n_H}{n_p} = \frac{n_p h^3}{(2\pi k_{\rm B} T m_e)^{3/2}} \mathrm{e}^{\beta E_d} \, .$$

- We define

$$X \equiv \frac{n_p}{n_p + n_H} \, , \;\; Z \equiv \frac{n_H}{n_p} \;\; \text{so } X = 1/(Z+1) \, .$$

- We also define the temperature of the recombination, T_R, when $Z = 1$ or $X = 1/2$.

- The known interactions among particles do not violate the baryon number.
- Since the co-moving entropy is also conserved, then n_B/s is constant.
- As $T \leq 0.5$ MeV, $s \propto n_\gamma$, therefore

$$\boxed{\frac{n_B}{n_\gamma} = \text{const}}$$

- We recall
  1. $n_\gamma = 20.28\, T^3$ photon/cm^3.
  2. Today, $T_0 = 2.725$ K, so $n_{\gamma,0} = 4.1 \times 10^8$ photon/m^3.
  3. $\rho_{c,0} = 3H_0^2/(8\pi G) = 1.88 \times 10^{-26} h^2$ kg/m^3. Remember $H_0 = 100h$ km/(s Mpc).
  4. $\Omega_B \equiv \rho_{B,0}/\rho_{c,0}$.
  5. We assume all the baryons are protons [But 10% of the matter are Helium atoms with neutrons!].
- $n_B = \rho_B/m_p = 11.3(\Omega_B h^2)$ baryon/m^3.
- $\eta \equiv n_B/n_\gamma = 11.3(\Omega_B h^2)/(4.1 \times 10^8) = 2.77 \times 10^{-8}(\Omega_B h^2)$.
- Using $\Omega_B h^2 = 0.0226$ (obs) then $\boxed{\eta = 6.08 \times 10^{-10}}$ which is constant!

- Remember, the equation to solve ($Z = 1$ at $T = T_R$) is

$$1 = \frac{n_p h^3}{(2\pi k_{\mathrm{B}} T_R m_e)^{3/2}} \mathrm{e}^{E_d/(k_{\mathrm{B}} T_R)} ,$$

  with $n_p = \frac{1}{2}\eta n_\gamma$ [Remember $Z = 1$].
- The solution is $T_R \simeq 0.32$ eV and $1 + z_R = T_R/T_0 = 1493$. Approx. 300 000 years after the BB.

More

We define

$$X_e = \frac{n_e}{n_B} = \frac{n_p}{n_B} ,$$

then

$$\frac{1 - X_e}{X_e^2} = \frac{(n_B - n_p)n_B}{n_p^2} = \frac{n_H n_B}{n_p^2} ,$$

where we have assumed that all the barions are protons and $n_B = n_p + n_H$.
Therefore

$$\frac{1 - X_e}{X_e^2} = \frac{n_B h^3}{(2\pi k_B T m_e)^{3/2}} e^{E_d/(k_B T)} ,$$

or

$$\frac{1 - X_e}{X_e^2} = \frac{\eta n_\gamma h^3}{(2\pi k_B T m_e)^{3/2}} e^{E_d/(k_B T)} .$$

Another definition of T_R is $X(T_R) = 1/10$, solving the equation, we obtain $T_R = 0.298$ eV.

[Remember: This is a good approximation assuming equilibrium, which is not fully true!].

Decoupling of photons

- It occurred slightly after the recombination.
- Photons are in thermal equilibrium via reactions

$$e^{\pm} + \gamma \leftrightarrow e^{\pm} + \gamma \,.$$

- The decoupling temperature is defined via

$$\Gamma(T_d) = H(T_d) \,,$$

where $\Gamma \simeq n_e \sigma_T c$. $\sigma_T = (8\pi/3)\left(e^2/(4\pi\epsilon_0 m_e c^2)\right)^2$ is the Thomson cross section.

- In general

$$\frac{\dot{a}}{a} = H(t) = H(T) = H_0 \left(\Omega_V + \Omega_M (a_0/a)^3 + \Omega_R (a_0/a)^4 + \Omega_k (a_0/a)^2\right)^{1/2}$$

- The universe is matter-dominated, so

$$H(T) = H_0 \sqrt{\Omega_M} \left(\frac{T}{T_0}\right)^{3/2} \,,$$

where we have used $a_0/a = T/T_0$. All the T's are the temperature of the photons.

- By using $n_e = X_e n_B$ and $n_B = \eta n_\gamma$, we can write

$$H_0 \sqrt{\Omega_M} \left(\frac{T_d}{T_0}\right)^{3/2} = X_e(T_d)\eta n_\gamma(T_d)\sigma_T c\,.$$

- $\Omega_M = 0.31$ (obs) and $\sigma_T = 6.7 \times 10^{-29}$ m^2, we obtain

$$X_e^{2/3}(T_d)T_d \simeq 9.2 \times 10^{-3}\,\mathrm{eV}$$

- Remember that X_e satisfies the Saha equation

$$\frac{1 - X_e}{X_e^2} = 7 \times 10^{-8} \exp(13.6/T(\mathrm{eV}))\,.$$

- The solution of both equations is $T_d \simeq 0.27$ eV, so $X(T_d) = 0.0063$ and $1 + z_d = T_d/T_0 \simeq 1000$.