Mercury's Perihelion Precession
(Gravitation and Cosmology)

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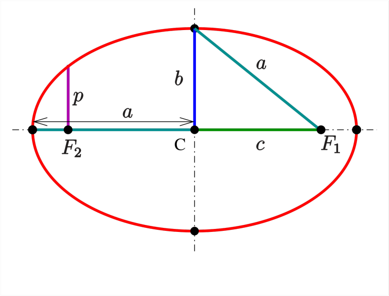
Badajoz, March 31th, 2023

Some properties of ellipses



where

- a: semi-major axis,
- b: semi-minor axis,
- $c = \epsilon a$: linear eccentricity,
- $p = a(1 - \epsilon^2)$: semi-latus rectum
- Polar equation centered in one of the foci:

$$r(\phi) = \frac{p}{1 + \epsilon \cos \phi} \, .$$

The Kepler problem

- Using the effective potential, we can write

$$\hat{e} = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{l^2}{2r^2} + V(r)\,,$$

  where $\hat{e} \equiv E/m$, $l \equiv L/m$, $V(r) \equiv U(r)/m$.

- Therefore

$$\left(\frac{dr}{dt}\right)^2 = 2\left(\hat{e} - \frac{l^2}{2r^2} - V(r)\right)\,.$$

- We perform the following change of variables $u = GM/r$, obtaining

$$\left(\frac{du}{d\phi}\right)^2 = \frac{2}{\bar{l}^2}\left(\hat{e} - \frac{\bar{l}^2}{2}u^2 - \hat{V}(u)\right)\,. \tag{1}$$

  where $\bar{l} \equiv l/(GM)$, $\hat{V}(u) \equiv V(GM/r)$ and we have used $l = r^2\frac{d\phi}{dt}$.

- In the Kepler problem: $V(r) = -GM/r$, so $\hat{V}(u) = -u$.
- Taking a derivative with respect to ϕ in Eq. (1), we finally obtain

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{\bar{l}^2}\,.$$

- The solution is

$$u(\phi) = \frac{1}{\bar{l}^2} + A\cos(\phi - \phi_0)\,. \tag{2}$$

  Hereafter we will consider ϕ_0 as the origin of the ϕ-angle, then $\phi_0 = 0$.
- So,

$$r(\phi) = \frac{GM\bar{l}^2}{1 + A\bar{l}^2\cos\phi}\,. \tag{3}$$

- By comparing with the equation of an ellipse

$$r(\phi) = \frac{p}{1 + \epsilon\cos\phi}\,.$$

  We can identify: $p = GM\bar{l}^2$ and $\epsilon = A\bar{l}^2$. To compute A, one substitutes Eq. (2) into Eq. (1), obtaining $A^2\bar{l}^4 = 2\hat{e}\bar{l}^2 + 1$.

Summarizing, given the energy $\hat{e} < 0$ and the angular momentum (both per unit mass of the orbiting particle) of the orbit we can compute the two elements of the ellipse (remember $\hat{e} < 0$):

$$a = \frac{GM}{2|\hat{e}|} \ ,$$

$$1 - \epsilon^2 = 2|\hat{e}|\bar{l}^2 \ .$$

General Relativity Computation

- The starting point are the equations

$$\mathcal{E} = \frac{1}{2c^2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r)\,, \tag{4}$$

$$V_{\text{eff}}(r) = \frac{1}{c^2}\left(-\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{c^2r^3}\right)\,, \tag{5}$$

where $\mathcal{E} \equiv \frac{e^2-1}{2}$.

- Using $l = r^2\frac{d\phi}{d\tau}$ (equatorial orbit) one can write

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{c^2r^4}{l^2}\left(e^2 - 1 + \frac{2GM}{c^2r} - \frac{l^2}{c^2r^2} + \frac{2GMl^2}{c^4r^3}\right)\,. \tag{6}$$

- And making use of $u = GM/r$ (as in the Kepler problem), we obtain

$$\left(\frac{du}{d\phi}\right)^2 = \frac{c^2}{\bar{l}^2}\left(e^2 - 1 + \frac{2u}{c^2} - \frac{\bar{l}^2u^2}{c^2} + \frac{2\bar{l}^2u^3}{c^4}\right)\,, \tag{7}$$

where, as usual, $\bar{l} \equiv l/(GM)$.

- Taking a derivative with respect to ϕ in Eq. (7), we finally obtain

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{\bar{l}^2} + \frac{3u^2}{c^2} \,. \tag{8}$$

- Dimensional check: $[u] = [c^2] = L^2T^{-2}$, $\bar{l} = L^{-1}T$.
- We can solve Eq. (8) perturbatively by assuming

$$u(\phi) = u_0 + \frac{1}{c^2}u_1 + O(\frac{1}{c^4}) \,. \tag{9}$$

- Plugging Eq. (9) into Eq. (8) and working consistently in $O(1/c^2)$ we obtain the following two equations

$$u_0'' + u_0 = \frac{1}{\bar{l}^2} \,, \tag{10}$$

$$u_1'' + u_1 = 3u_0^2 \,, \tag{11}$$

where $f' \equiv df/d\phi$.

- Eq. (10) is just the Kepler equation. The solution is

$$u_0 = \frac{1}{\bar{l}^2}\left(1 + \epsilon \cos\phi\right)$$

- Plugging u_0 into Eq. (11) we obtain

$$u_1'' + u_1 = \frac{3}{\bar{l}^4}\left(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi\right)$$

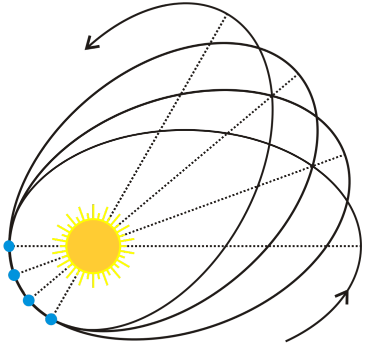
- Finally the solution is

$$u = \frac{1}{\bar{l}^2}\left(1 + \epsilon\cos\phi\right) + \frac{3}{c^2\bar{l}^4} + \frac{3\epsilon}{c^2\bar{l}^4}\phi\sin\phi + \frac{3\epsilon^2}{2c^2\bar{l}^4} - \frac{\epsilon^2}{2c^2\bar{l}^4}\cos 2\phi + O\left(\frac{1}{c^4}\right).$$

- The only unbounded term is the secular one (in red), the remaining $O(1/c^2)$ terms are bounded. Considering only the secular term, we can finally write

$$u \simeq \frac{1}{\bar{l}^2}(1+\epsilon\cos\phi) + \frac{3\epsilon}{c^2\bar{l}^4}\phi\sin\phi \simeq \frac{1}{\bar{l}^2}\left\{1+\epsilon\cos\left(\phi - \frac{3}{c^2\bar{l}^2}\phi\right)\right\},$$

which is the equation of an ellipse which precesses.
- The shape of this orbit is like



- This orbit comes back to a given value of u when

$$\phi_{u=\text{const}}\left(1-\frac{3}{c^2\bar{l}^2}\right)=2\pi\,,$$

and the motion of the Mercury's perihelion, per revolution, is

$$\Delta\phi=\phi_{u=\text{const}}-2\pi\simeq\frac{6\pi}{c^2\bar{l}^2}\,.$$

- Recalling from the Kepler problem that $\bar{l}^2=a(1-\epsilon^2)/(GM)$ we get

$$\boxed{\Delta\phi=\frac{6\pi GM}{ac^2(1-\epsilon^2)}\,.}\tag{12}$$

Applying this formula to the Mercury (☿) orbit:

- Data: $M_\odot = 2 \times 10^{30}$ kg, $a_☿ = 5.8 \times 10^{10}$ m, $\epsilon_☿ = 0.21$.
- Using the '1-2-3' law we can compute the period of the Mercury's orbit:

$$GM_\odot = \omega^2 a_☿{}^3 \,.$$

  - $T_☿ = 0.24$ year.
- $\Delta\phi_☿ = 0.104''$/rev.
- Then $\Delta\phi_☿ = 43.3''$/century. Excellent agreement!!!