Homework. Gravitation and Cosmology. Year 2022/2023

1. A Killing vector ξ satisfies $\mathcal{L}_\xi \boldsymbol{g} = 0$, where $\mathcal{L}_\xi$ is the Lie-derivative along the vector ξ and $\boldsymbol{g}$ is the metric. Show that

$$\xi_{(\alpha;\beta)} \equiv \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0\,.$$

2. Consider the following two-dimensional metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2 + r^2\mathrm{d}\theta^2}{(1+r^2)^2}\,,$$

   where $r \in \mathbb{R}^+$ and $\theta \in [0, 2\pi)$.

   - Compute the equations of the geodesics as a function of the arc-length s and the associated Christoffel's symbols (Lagrangian method).
   - Compute the Ricci curvature scalar R of this space.
   - Compute explicitely a Killing vector and check it satisfies the equation $\xi_{(i;j)} = 0$.
   - Is flat this space? Justify the answer.

3. Compute the modulus of the four-acceleration ($\boldsymbol{a}$) of an static observer (r, θ and ϕ constants) in the following metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = -f(r)\mathrm{d}t^2 + 2\mathrm{d}t\mathrm{d}r + r^2\mathrm{d}\Omega_2^2\,,$$

   where $\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

   Note: The four-acceleration is defined as $\boldsymbol{a} = \nabla_{\boldsymbol{u}}\boldsymbol{u}$ where $\boldsymbol{u}$ is the four-velocity of the static observer.

4. Show

$$V_{\alpha;\nu\kappa} - V_{\alpha;\kappa\nu} = V_\sigma R^\sigma{}_{\alpha\nu\kappa}\,.$$

5. Show

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}V^\mu)\,,$$

   where $g \equiv -\det(g_{\mu\nu})$.

6. Consider the metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = \mathrm{d}u^2 - u^2\mathrm{d}v^2\,.$$

   Compute:

   - The differential equations of the geodesics: $u(\tau)$ and $v(\tau)$.
   - The Christoffel's symbols (Lagrangian method).
   - The differential equation of the geodesic $u(v)$.
   - Is plane this space?
   - A Killing vector and its associated conserved quantity.
   - Show that this Killing vector satisfies the following equations

$$\nabla_i\xi_j + \nabla_j\xi_i = 0\,.$$

7. Consider the four velocity, $\boldsymbol{u}$, of a free falling observer in a Schwarzschild geometry. Show, by computing explicitly the covariant derivative, that

$$\nabla_{\boldsymbol{u}}\boldsymbol{u} = 0\,.$$

8. Consider the following stationary and spherically symmetric metric

$$ds^2 = -d\tau^2 = -f_1(r)dt^2 + \frac{1}{f_1(r)}dr^2 + f_2^2(r)d\Omega_2^2\,,$$

with $d\Omega_2^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$.

- Compute two Killing vectors, related with t and ϕ, and their associated conserved quantities (denoted by e and l, respectively)
- Show that the particles move in an equatorial orbit following the equation

  $$\dot{r}^2 + V(r) = e^2\,,$$

  con

  $$V(r) = f_1(r)\left(\epsilon + \frac{l^2}{f_2^2(r)}\right)\,,$$

  with $\epsilon = 0$ for massless particles ($\dot{r} = dr/d\lambda$, where λ is an affine parameter) or $\epsilon = 1$ for massive particles ($\dot{r} = dr/d\tau$).
- Determine the conditions the functions f_1 and f_2 should satisfy in order to have null geodesics with the coordinate r constant.

9. Find the function $f(v)$ so that the metric

$$ds^2 = 2du dz + f(z)dy^2 + 2dy dx$$

is solution of the Einstein equations in the vacuum. Determine the conditions $f(z)$ should satisfy in order to have a plane space-time.

10. Consider two observers at rest in a gravitational field given by the following metric (anti-de Sitter space):

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{l^2}} + r^2 d\Omega_2^2\,,$$

l being a parameter with dimensions of length. The first observer has spatial coordinates $r_1 = a$, $\theta_1 = \pi/2$, $\phi_1 = 0$ and the second $r_2 = b$, $\theta_2 = \pi/2$, $\phi_2 = 0$, with $b \ll a \ll l$. The first observer emits light which propagates in the direction $\theta = \pi/2$ and $\phi = 0$. This light is detected by the second observer. What is the redshift as seen by the second observer (blue or red)?

11. The metric of Reissner-Nordstrøm

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2^2\,,$$

is a solution of the Einstein equations with $\Lambda = 0$ and the energy-momentum tensor produced by a electromagnetic field. This metric describes the external gravitational field of a charged star or a static charged black hole. The parameters M and Q are related with the mass and charge of the star or black hole. We will assume hereafter that $M^2 > Q^2$, why?

- Show that a chargeless massive particle falling radially can never reach $r = 0$. Compare this situation with a particle falling in the Schwarzschild metric.
- Consider a particle initially is at rest at infinity. Compute the minimum radial coordinate of the particle during its falling, $r_{\min}$. Check that $r_{\min} < r_-$, where r_- is the smallest root of the equation

  $$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0\,.$$

12. Consider a Schwarzschild black hole with mass M.

- Show that the light can follow a circular orbit with radial coordinate $r = 3M$.
- Discuss the stability of this orbit.
- Compute in coordinate time the period of this orbit.
- Compute the period measured by an observer at rest in the infinity.
- Consider an observer at rest in a fixed point of the orbit $r = 3M$. Compute the period measured by this observer?

13. Show ($K = 0$):

$$\ddot{z} = \frac{\dot{z}^2}{1+z}\left(\frac{5}{2} + \frac{3p}{2\rho}\right) .$$

14. Consider a universe with $h = 0.7$, $\Omega_\Lambda = 0.55$, $\Omega_M = 0.45$ and $\Omega_R = 0$.

- Compute the age of the universe.
- Compute its curvature.
- Compute $m - M$ and the luminosity distance (d_L) for a star with $z = 4$? How long was the light of this star emitted?

15. Show:

$$\Omega_k(z) = \frac{\Omega_k}{\Omega_M(1+z) + \Omega_R(1+z)^2 + \Omega_\Lambda(1+z)^{-2} + \Omega_k} .$$

16. Show

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(1 - \Omega(t)\right) = -2\frac{\ddot{a}}{\dot{a}}\left(1 - \Omega(t)\right) ,$$

where

$$\Omega(t) \equiv \Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t) ,$$

$$\Omega_i \equiv \frac{\rho_i(t)}{\rho_c(t)} , \; i = R, M, \Lambda ,$$

$$\rho_c(t) \equiv \frac{3H^2(t)}{8\pi G} .$$

$\dot{a} \equiv \mathrm{d}a/\mathrm{d}t$.

17. Suppose that the scale factor $a(t)$ verifies

$$a(t) = (t/t_*)^{1/2} ,$$

where t_* is a constant and t is the proper time elapsed since the singularity. Also suppose that the age of the universe is 13.8 billion of years.

- Compute $H_0 = H(t_0)$ (t_0 meaning today).
- Compute the age of the Universe at which $T_\gamma = 1000$ K. Note $T_{\gamma,0} = 2.7$ K.

18. Consider a flat universe and neglect the radiation. A galaxy has been observed with $z = 1.55$ and with an age of 3.5×10^9 years (in the moment of the emission of the light). Find a lower bound for Ω_Λ. Assume $H_0 = 70$ km/s/Mpc.