Additional Homeworks. Gravitation and Cosmology. Year 2020/2021

1. A Killing vector ξ satisfies $\mathcal{L}_\xi \boldsymbol{g} = 0$, where $\mathcal{L}_\xi$ is the Lie-derivative along the vector ξ and $\boldsymbol{g}$ is the metric. Show that

$$\xi_{(\alpha;\beta)} = 0\,.$$

2.
   - Let χ^μ be a vector satisfying

   $$\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = c g_{\mu\nu}$$

   being c a positive constant and let γ be a null geodesic with tangent vector u^μ. Show that $u^\mu \chi_\mu$ is constant along the geodesic γ.
   - Let $T_{\mu\nu}$ and ξ be a symmetric conserved tensor ($\nabla_\mu T^{\mu\nu} = 0$) and a Killing vector respectively. Show that the current defined as $J^\mu \equiv T^{\mu\nu}\xi_\nu$ is conserved ($\nabla_\mu J^\mu = 0$).

3. Consider the following metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{1+x^2} + x^2 \mathrm{d}y^2\,.$$

   Compute:

   - The differential equations of the geodesics: $y(s)$ and $x(s)$.
   - The Christoffel's symbols (Lagrangian method).
   - The geodesic with initial condition: $y(0) = 0$, $x(0) = b$, $\dot{y}(0) = 0$ y $\dot{x}(0) = 0$. ($\dot{f} = \mathrm{d}f/\mathrm{d}s$).
   - Is plane this space?
   - A Killing vector and its associated conserved quantity.
   - Show that the vector

   $$\boldsymbol{w} = \sqrt{1+x^2}\cos(y)\partial_x - \frac{\sqrt{1+x^2}}{x}\sin(y)\partial_y$$

   is a Killing one. Show that it satisfies the following equations

   $$\nabla_i w_j + \nabla_j w_i = 0\,.$$

4. Consider the metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = -w^2 \mathrm{d}y^2 + \mathrm{d}w^2\,.$$

   Compute:

   - The differential equations of the geodesics: $w(\tau)$ and $y(\tau)$.
   - The Christoffel's symbols (Lagrangian method).
   - The differential equation of the geodesic $w(y)$.
   - Is plane this space?
   - A Killing vector and its associated conserved quantity.
   - Show that this Killing vector satisfies the following equations

   $$\nabla_i w_j + \nabla_j w_i = 0\,.$$

5. Consider the metric on the spehere S^2:

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2\,.$$

   Show that the following three vectors

$$\begin{aligned}\boldsymbol{v}_1 &= \partial_\phi \\ \boldsymbol{v}_2 &= \cos\phi\cot\theta\ \partial_\phi + \sin\phi\ \partial_\theta \\ \boldsymbol{v}_3 &= -\sin\phi\cot\theta\ \partial_\phi + \cos\phi\ \partial_\theta\end{aligned}$$

   are Killing vectors. What Lie algebra do they generate?

Additional Homeworks (II). Gravitation and Cosmology. Year 2020/2021

1. Find the function $f(v)$ so that the metric

$$ds^2 = 2dudv + f(v)dw^2 + 2dwdz$$

is solution of the Einstein equations in the vacuum. Determine the conditions $f(v)$ should satisfy in order to have a plane space-time.

2. Consider two observers at rest in a gravitational field given by the following metric (anti-de Sitter space):

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2}} + r^2 d\Omega_2^2 \,,$$

l being a parameter with dimensions of length. The first observer has spatial coordinates $r_1 = a$, $\theta_1 = \pi/2$, $\phi_1 = 0$ and the second $r_2 = b$, $\theta_2 = \pi/2$, $\phi_2 = 0$, with $b \ll a \ll l$. The first observer emits light which propagates in the direction $\theta = \pi/2$ and $\phi = 0$. This light is detected by the second observer. What is the redshift as seen by the second observer (blue or red)?

3. The metric of Reissner-Nordstrøm

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \,,$$

is a solution of the Einstein equations with $\Lambda = 0$ and the energy-momentum tensor produced by a electromagnetic field. This metric describes the external gravitational field of a charged star or a static charged black hole. The parameters M and Q are related with the mass and charge of the star or black hole. We will assume hereafter that $M^2 > Q^2$, why?

   - Show that a chargeless massive particle falling radially can never reach $r = 0$. Compare this situation with a particle falling in the Schwarzschild metric.
   - Consider a particle initially is at rest at infinity. Compute the minimum radial coordinate of the particle during its falling, r_{min}. Check that $r_{\text{min}} < r_-$, where r_- is the smallest root of the equation

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0 \,.$$

4. Consider a Schwarzschild black hole with mass M.

   - Show that the light can follow a circular orbit with radial coordinate $r = 3M$.
   - Discuss the stability of this orbit.
   - Compute in coordinate time the period of this orbit.
   - Compute the period measured by an observer at rest in the infinity.
   - Consider an observer at rest in a fixed point of the orbit $r = 3M$: compute the period measured by this observer?

5. Consider a $4 + 1$ space-time.

   - What is the Newtonian gravitational potential?
   - Show that the metric

$$ds^2 = -\left(1 - \frac{r_s^2}{r^2}\right) dt^2 + \left(1 - \frac{r_s^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2$$

   is a solution of the Einstein equations in the vacuum (is the $4 + 1$ analogous of the Schwarzschild solution). Moreover, $d\Omega_3^2$ is the metric of the 3-sphere S^3 and r_s is a constant.

Additional Homeworks (III). Gravitation and Cosmology. Year 2020/2021

1. Consider a homogeneous, isotropic, cosmological model described by the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(\frac{t}{t_*}\right)^2 \left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] ,$$

   where t_* is a constant.

   (a) Is this model open, closed, or flat?

   (b) Is this a matter-dominated universe? Explain.

   (c) Assuming the Friedmann equations holds for this universe, find $\rho(t)$.

2. Build the three possible maximally symmetric spaces in 1+3 dimensions.

   Help: You need to build the Minkowski, de Sitter and anti-de Sitter spaces.

3. Consider a universe with only matter.

   - Compute $a(t)$, $t(z)$ y $d_A(z)$ as a function of H_0^{-1}.
   - Compute the age of the universe today and when the universe had $z = 1000$. Take $h = 0.68$.
   - Compute $d_A(z)$ as $z = 100$.
   - Show that the function $d_A(z)$ presents a maximum and compute the value of z at the maximum.

   Note: $d_A(z) = d_L/(1+z)^2$, where d_L is the luminosity-distance.

4. Consider a universe with only matter and cosmological constant $\Lambda \equiv 8\pi G\rho_\Lambda > 0$. Suppose that this is an static universe ($a(t)$ does not depend on t).

   - Compute ρ_Λ as a function of ρ_M.
   - Compute the curvature constant of the Robertson-Walker metric (K).
   - Compute the volume of this universe as a function of Λ.

5. Write an integral, as function of H_0 and the Ω's, to compute the proper distance to the visible horizon. What is this distance today (light-years)? Take: $h = 0.68$, $\Omega_M = 0.31$, $\Omega_\Lambda = 0.69$ and $\Omega_R = 0$.

   Help: Evaluate numerically the integral.