Friedmann Equations
(Gravitation and Cosmology)

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- The starting point is the use of the Einstein's equations

$$R_{\mu\nu} = S_{\mu\nu} \,,$$

where

$$S_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\rho}{}^{\rho} \right) .$$

and the Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right) .$$

- Assuming the energy-momentum tensor for an ideal fluid

$$T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) u_\mu u_\nu \,.$$

and a co-moving observer

$$u^\mu = (1, 0, 0, 0) \,,$$

we obtain

$$T \equiv T_\mu{}^\mu = p g_\mu{}^\mu + (p + g) u^2 = 4p - (\rho + p) = 3p - \rho \,.$$

- Using the metric and u^μ we can compute the tensor $S_{\mu\nu}$, the nonzero components are

$$S_{tt} = 4\pi G\,(3p+\rho)\ ,$$

$$S_{rr} = 4\pi G\,(p-\rho)\,\frac{a^2(t)}{1-Kr^2}\,,$$

$$S_{\theta\theta} = 4\pi G\,(p-\rho)\,a^2(t)r^2\,,$$

$$S_{\phi\phi} = 4\pi G\,(p-\rho)\,a^2(t)r^2\sin^2\theta\,.$$

- Using Mathematica or SageManifolds we can write the following nonzero components of the Ricci tensor [$\dot{a} \equiv \mathrm{d}a/\mathrm{d}t$]:

$$R_{tt} = -3\ddot{a}/a\,,$$

$$R_{rr} = \frac{2K + 2\dot{a}^2 + a\ddot{a}}{1-Kr^2}\ ,$$

$$R_{\theta\theta} = r^2\left(2(K+\dot{a}^2) + a\ddot{a}\right)\,,$$

$$R_{\phi\phi} = r^2\sin^2\theta\left(2(K+\dot{a}^2) + a\ddot{a}\right)\,.$$

Therefore one can write the Einstein's equations as

- t:

$$\boxed{-3\frac{\ddot{a}}{a} = 4\pi G(3p + \rho)}$$

- r, θ, ϕ:

$$\boxed{2K + 2\dot{a}^2 + a\ddot{a} = 4\pi G(\rho - p)a^2}$$

And finally:

$$\boxed{\boxed{\dot{a}^2 + K = \frac{8\pi G}{3}\rho a^2}}$$

The second ingredient to complete the Friedmann equations is to use the conservation of the energy-momentum tensor:

$$\nabla_\beta T^{\alpha\beta} = 0 \,.$$

The component t of this conservation law is written as

$$\frac{\partial T^{t\beta}}{\partial x^\beta} + \Gamma^t_{\beta\gamma} T^{\gamma\beta} + \Gamma^\beta_{\beta\gamma} T^{t\gamma} = 0 \,.$$

Notice that $T^{\alpha\beta}$ is diagonal and $T^{tt} = \rho$.

- $$\frac{\partial T^{t\beta}}{\partial x^\beta} = \frac{\partial T^{tt}}{\partial t} = \dot{\rho}$$
$$T^{rr} = p g^{rr} = p(1 - Kr^2)/a^2$$
$$T^{\theta\theta} = p g^{\theta\theta} = p/(a^2 r^2)$$
$$T^{\phi\phi} = p g^{\phi\phi} = p/(a^2 r^2 \sin^2\theta)$$
- Using the nonzero Christoffel symbols (SageManifolds), we finally get:

$$\boxed{\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0}$$

The component r of this conservation law is written as

$$\frac{\partial T^{r\beta}}{\partial x^{\beta}} + \Gamma^{r}_{\beta\gamma} T^{\gamma\beta} + \Gamma^{\beta}_{\beta\gamma} T^{r\gamma} = 0$$

$$\frac{\partial T^{rr}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{p(1-Kr^2)}{a^2}\right) = -2pKr/a^2$$

obtaining

$$0 = 0$$

and the same for the components θ and ϕ.

Cosmological Constant (Vacuum Energy)

The energy-momentum tensor of the vacuum must be written as (using Lorentz invariance)

$$T^{\mu\nu}_{\mathrm{vac}} \propto g^{\mu\nu}$$

Assuming the usual form of this tensor for an ideal fluid

$$T_{\mu\nu} = p g_{\mu\nu} + (p+\rho) u_\mu u_\nu$$

then

$$\boxed{p = -\rho}$$

Another derivation.

- ρ is positive and constant.
- First principle of Thermodynamics and adiabatic evolution:

$$0 = \mathrm{d}U + p\mathrm{d}V = \mathrm{d}(\rho a^3) + p\mathrm{d}a^3$$

- Therefore, $\rho = -p$.

Behavior of the matter and energy: The equation of state

Consider one of the Friedmann equations

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0$$

and the equation of state (which defines ω)

$$p = \omega\rho$$

obtaining

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}$$

whose solution is

$$\rho \sim a^{-3(1+\omega)}$$

Examples:

- Cold matter (dust): $p = 0$ and then $\omega = 0$, so
$$\rho \sim a^{-3}$$
- Hot matter (e.g. ultrarelativistic particles, radiation): $p = \rho/3$, so $\omega = 1/3$, then
$$\rho \sim a^{-4}$$
- Vacuum energy (Cosmological constant): $p = -\rho$, then $\omega = -1$ and
$$\rho \sim \text{const}$$