Robertson-Walker metric
(Gravitation and Cosmology)

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Cosmological Principle

- The constant cosmic time hypersurfaces of the universe are homogeneous and isotropic.
- They must be described by a maximally symmetric space (maximal number of Killing vectors).
- The maximally symmetric spaces are spaces of constant curvarture.
- Hence, we need to build three models with positive, zero and negative constant curvature. Via change of variables (diffeomorphisms), they will describe all the constant cosmic time hypersurfaces of the universe.
- We need to build 3-dimensional constant curvature spaces.

- Zero curvature ($3D$ Euclidean):

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 \equiv \mathrm{d}\boldsymbol{x}^2 \,.$$

- Positive curvature (S^3)

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 + \mathrm{d}u^2 = \mathrm{d}\boldsymbol{x}^2 + \mathrm{d}u^2 \,,$$

  with the constraint

$$a^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + u^2 = \boldsymbol{x}^2 + u^2 \,.$$

- Negative curvature (H^3)

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 - \mathrm{d}u^2 = \mathrm{d}\boldsymbol{x}^2 - \mathrm{d}u^2 \,,$$

  with the constraint

$$a^2 = u^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = u^2 - \boldsymbol{x}^2 \,.$$

- We can write S^3 and H^3 in a more compact way

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x}^2 \pm \mathrm{d}u^2 ,$$

$$a^2 = u^2 \pm \boldsymbol{x}^2 .$$

- We rescale the variables as $\boldsymbol{x} = a\boldsymbol{x}'$ and $u = au'$, and then dropping the primes, so:

$$\mathrm{d}s^2 = a^2 \left(\mathrm{d}\boldsymbol{x}^2 \pm \mathrm{d}u^2\right) ,$$

$$1 = u^2 \pm \boldsymbol{x}^2 .$$

- Differentiating the last equation

$$2u\mathrm{d}u \pm 2\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} = 0$$

and then

$$\mathrm{d}u = \mp \frac{\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x}}{u}$$

- Therefore

$$\mathrm{d}s^2 = a^2\left(\mathrm{d}\boldsymbol{x}^2 \pm \frac{(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x})^2}{u^2}\right)$$

and

$$\mathrm{d}s^2 = a^2\left(\mathrm{d}\boldsymbol{x}^2 \pm \frac{(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x})^2}{1 \mp \boldsymbol{x}^2}\right).$$

- By introducing $K = 0$ (Euclidean), $K = 1$ (S^3) and $K = -1$ (H^3) and adding the time, we can finally obtain the Robertson-Walker metric

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}\boldsymbol{x}^2 + \frac{K(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x})^2}{1 - K\boldsymbol{x}^2}\right)}.$$

Note: It is possible to remove the terms g_{0j} of the metric.

- Theorem. This metric is unique (modulo diffeomorphisms) if the universe appears spherically symmetric and isotropic to a set of observers in free fall, such as the astronomers of typical galaxies.

- Using sphericall coordinates $\boldsymbol{x} = r\boldsymbol{e}_r$ and so $\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} = r\mathrm{d}r$ and the RW metric reads as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right).$$

  [Remember: $\mathrm{d}\boldsymbol{x}^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$.]

- By defining

$$\mathrm{d}\chi^2 \equiv \frac{\mathrm{d}r^2}{1 - Kr^2},$$

  then

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\chi^2 + S_K^2(\chi)\mathrm{d}\Omega_2^2,$$

  where

$$S_K(\chi) \equiv \mathrm{sinn}(\chi) = \begin{cases} \sin(\chi)\,, & K = 1\,, \\ \chi\,, & K = 0\,, \\ \sinh(\chi)\,, & K = -1\,. \end{cases}$$

Co-moving coordinates

- Verify $\Gamma^i_{00} = 0$.
- Remember, the geodesic equation for a spatial coordinate is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + \Gamma^i_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0\,.$$

- And $x^i = \text{const}$ is solution.
- For a co-moving observer $x^i = \text{const}$ and $\mathrm{d}\tau = \mathrm{d}t$. Therefore t is the proper time of any co-moving observer.