SOLUTIONS OF THE MOMENT HIERARCHY IN THE KINETIC THEORY OF MAXWELL MODELS

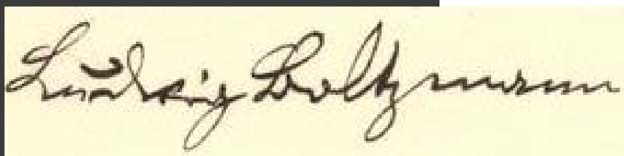
Andrés Santos
Universidad de Extremadura, Badajoz (Spain)

STATISTICAL PHYSICS in EXTREMADURA

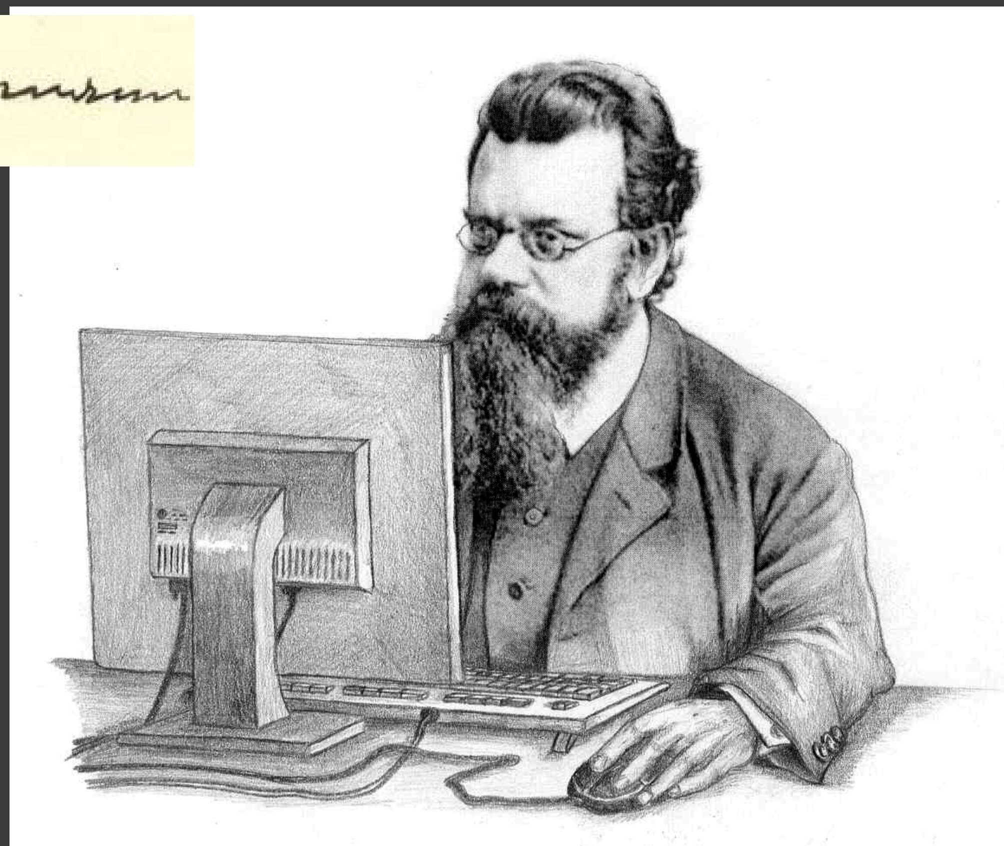
Outline

- Moment equations for Maxwell molecules
- Some solvable states:
  - Planar Fourier flow
  - Planar Fourier flow with gravity
  - Planar Couette flow
  - Force-driven Poiseuille flow
  - Uniform shear and longitudinal flows
- Conclusions

The Boltzmann equation



(1844-1906)



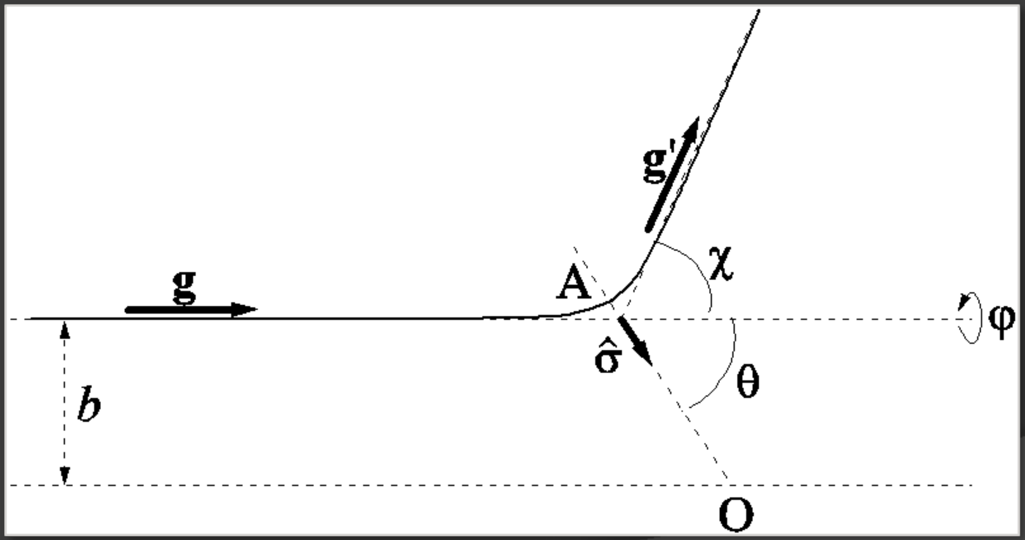
(Cartoon by Bernhard Reischl, University of Vienna)

$$\frac{\partial}{\partial t}f + \underbrace{\mathbf{v}\cdot\nabla f}_{\text{streaming}} + \underbrace{\frac{\partial}{\partial \mathbf{v}}\cdot\left(\frac{\mathbf{F}}{m}f\right)}_{\text{external force}} = \underbrace{J[\mathbf{v}|f,f]}_{\text{collisions}}$$

$$J[\mathbf{v}|f,f] \equiv \int d\mathbf{v}_1 \int d\Omega\, g \boxed{B(g,\chi)} \left[f(\mathbf{v}')f(\mathbf{v}'_1) - f(\mathbf{v})f(\mathbf{v}_1)\right]$$

Differential cross section

$$\mathbf{v}' = \mathbf{v} - (\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}, \quad \mathbf{v}'_1 = \mathbf{v}_1 + (\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}$$



$$B(g,\chi) = \frac{b}{\sin\chi}\left|\frac{db}{d\chi}\right|$$

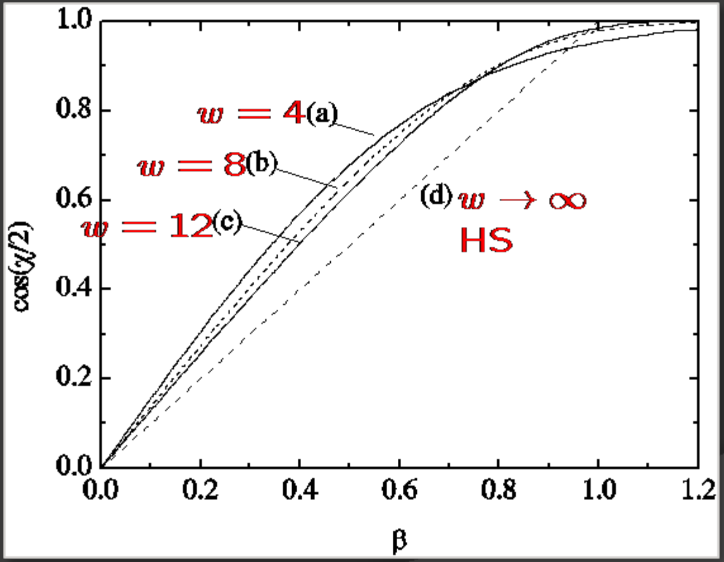
$$\chi(b,g) = \pi - 2\int_{r_0}^{\infty} dr \frac{b/r^2}{\left[1-(b/r)^2 - 4\boxed{\phi(r)}/mg^2\right]^{1/2}}$$

Interaction potential

$$\phi(r) \sim r^{-w}$$

$$\chi(b,g) = \chi(\beta), \quad \beta \sim bg^{2/w}$$

$$B(g,\chi) = g^{-4/w}\mathcal{B}(\chi)$$



Weak form of the Boltzmann equation

$$\underbrace{\Psi(\mathbf{r},t) = \int d\mathbf{v}\,\psi(\mathbf{v}) f(\mathbf{r},\mathbf{v};t)}_{\text{Density of } \psi(\mathbf{v})} \Rightarrow \boxed{\frac{\partial \Psi}{\partial t} + \nabla \cdot \mathbf{\Phi}_\psi = \sigma_\psi^{(F)} + J_\psi}$$

$$\underbrace{\mathbf{\Phi}_\psi = \int d\mathbf{v}\,\mathbf{v}\psi(\mathbf{v}) f(\mathbf{v}),}_{\text{Flux}} \qquad \underbrace{\sigma_\psi^{(F)} = \int d\mathbf{v}\,\frac{\partial \psi}{\partial \mathbf{v}} \cdot \frac{\mathbf{F}}{m} f(\mathbf{v})}_{\text{Source: external force}}$$

$$\underbrace{\begin{aligned} J_\psi &= \int d\mathbf{v}\,\psi(\mathbf{v}) J[\mathbf{v}|f,f] \\ &= \frac{1}{4}\int d\mathbf{v}\int d\mathbf{v}_1 \int d\Omega\, g B(g,\chi)\left[\psi(\mathbf{v}) + \psi(\mathbf{v}_1) - \psi(\mathbf{v}') - \psi(\mathbf{v}_1')\right] \\ &\quad \times \left[f(\mathbf{v}')f(\mathbf{v}_1') - f(\mathbf{v})f(\mathbf{v}_1)\right] \end{aligned}}_{\text{Source: collisions}}$$

Hierarchy of moment equations

$$\frac{\partial \Psi}{\partial t} + \nabla \cdot \mathbf{\Phi}_\psi = \sigma_\psi^{(F)} + J_\psi$$

$\psi(\mathbf{v}) =$ polynomial of degree $k \Rightarrow \Psi =$ velocity moment of degree k

$\sigma_\psi^{(F)} =$ velocity moment of degree $k-1$, if $\mathbf{F} \neq \mathbf{F}(\mathbf{v})$

$\mathbf{\Phi}_\psi =$ velocity moment of degree $\boxed{k+1}$

$J_\psi =$ bilinear combination of velocity moments of any degree

Maxwell molecules

On the Dynamical Theory of Gases
Phil. Trans. Roy. Soc. (London) **157**, 49-88 (1867)

(1831-1879)

In the present paper I propose to consider the molecules of a gas, not as elastic spheres of definite radius, but as small bodies or groups of smaller molecules repelling one another with a force whose direction always passes very nearly through the centres of gravity of the molecules, and whose magnitude is represented very nearly by some function of the distance of the centres of gravity.

I have made this modification of the theory in consequence of the results of my experiments on the viscosity of air at different temperatures, and I have deduced from these experiments that the repulsion is inversely as the *fifth* power of the distance.

I have found by experiment that the coefficient of viscosity in a given gas is independent of the density, and proportional to the absolute temperature, so that if ET be the viscosity, $ET \propto p/\rho$.

Maxwell molecules

$$\phi(r) \sim r^{-4} \Rightarrow gB(g,\chi) = \mathcal{B}(\chi)$$

True Maxwell potential (inverse power law, IPL):

$$\mathcal{B}(\chi) \propto \frac{\beta}{\sin\chi}\left|\frac{d\beta}{d\chi}\right|, \quad \chi(\beta) = \pi - 2\int_0^{\beta_0} d\beta' \left[1 - \beta'^2 - \frac{1}{2}\left(\frac{\beta'}{\beta}\right)^4\right]^{-1/2}$$

Variable soft-sphere (VSS) model: $\mathcal{B}(\chi) = \cos^{2(\alpha-1)}\frac{\chi}{2}, \quad \alpha = 2.13986$

Variable hard-sphere (VHS) model: $\mathcal{B}(\chi) = \text{const}, \quad \alpha = 1$

Properties of the collisional moments in Maxwell models

$$\text{If } \psi(\mathbf{v}) = \text{polynomial of degree } k$$

$$\text{then}$$

$$\begin{aligned} J_\psi \;&=\; \text{bilinear combination of velocity moments} \\ &\quad\;\; \text{of degree equal to or smaller than } k \end{aligned}$$

Properties of the collisional moments in Maxwell models. Eigenfunctions and eigenvalues

$$\underbrace{\psi_{r\ell\mu}(\mathbf{c}) \propto c^\ell L_r^{(\ell+\frac{1}{2})}(c^2) Y_\ell^\mu(\widehat{\mathbf{c}}),}_{\text{Eigenfunctions}} \quad k = 2r+\ell, \quad \mathbf{c} = (m/2k_B T)^{1/2}(\mathbf{v}-\mathbf{u})$$

$$\Psi \to \mathcal{M}_{r\ell\mu}$$

$$J_\psi \to \mathcal{J}_{r\ell\mu} = -\lambda_{r\ell}\mathcal{M}_{r\ell\mu} + \cdots$$

$$\underbrace{\begin{aligned}\lambda_{r\ell} \;=\; & n2\pi \int_0^\pi d\chi \sin\chi \mathcal{B}(\chi) \left[1 + \delta_{r0}\delta_{\ell 0} - \cos^{2r+\ell}\frac{\chi}{2} P_\ell\left(\cos\frac{\chi}{2}\right)\right. \\ & \left. - \sin^{2r+\ell}\frac{\chi}{2} P_\ell\left(\sin\frac{\chi}{2}\right)\right]\end{aligned}}_{\text{Eigenvalues}}$$

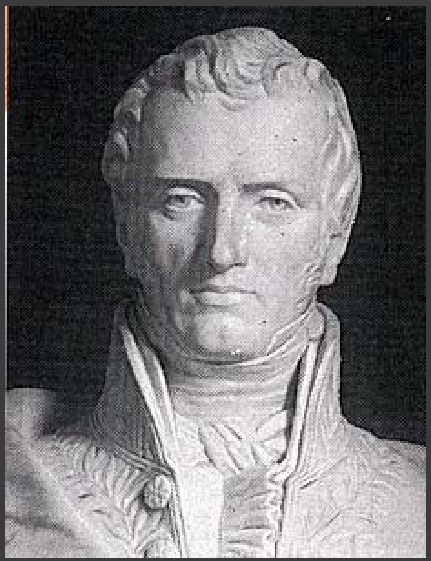
Properties of the eigenvalues in Maxwell models

$$\lambda_{r,1} = \lambda_{r+1,0}, \quad \bar{\lambda}_{02} = \frac{3}{2}, \quad \bar{\lambda}_{03} = \frac{9}{4}, \quad \bar{\lambda}_{12} = \frac{7}{4}, \quad \bar{\lambda}_{21} = \frac{3}{2}, \quad \bar{\lambda}_{r\ell} \equiv \frac{\lambda_{r\ell}}{\lambda_{11}}$$

VSS and VHS models:

$$\begin{aligned} \bar{\lambda}_{r\ell} \;&=\; (1+\alpha)(2+\alpha)\int_0^1 dx\, x^{2\alpha-1}\Big[1+\delta_{r0}\delta_{\ell 0} - x^{2r+\ell}P_\ell\,(x) \\ &\quad -(1-x^2)^{r+\ell/2}P_\ell\left(\sqrt{1-x^2}\right)\Big] \end{aligned}$$

Navier-Stokes constitutive equations



Claude-Louis Navier
(1785-1836)

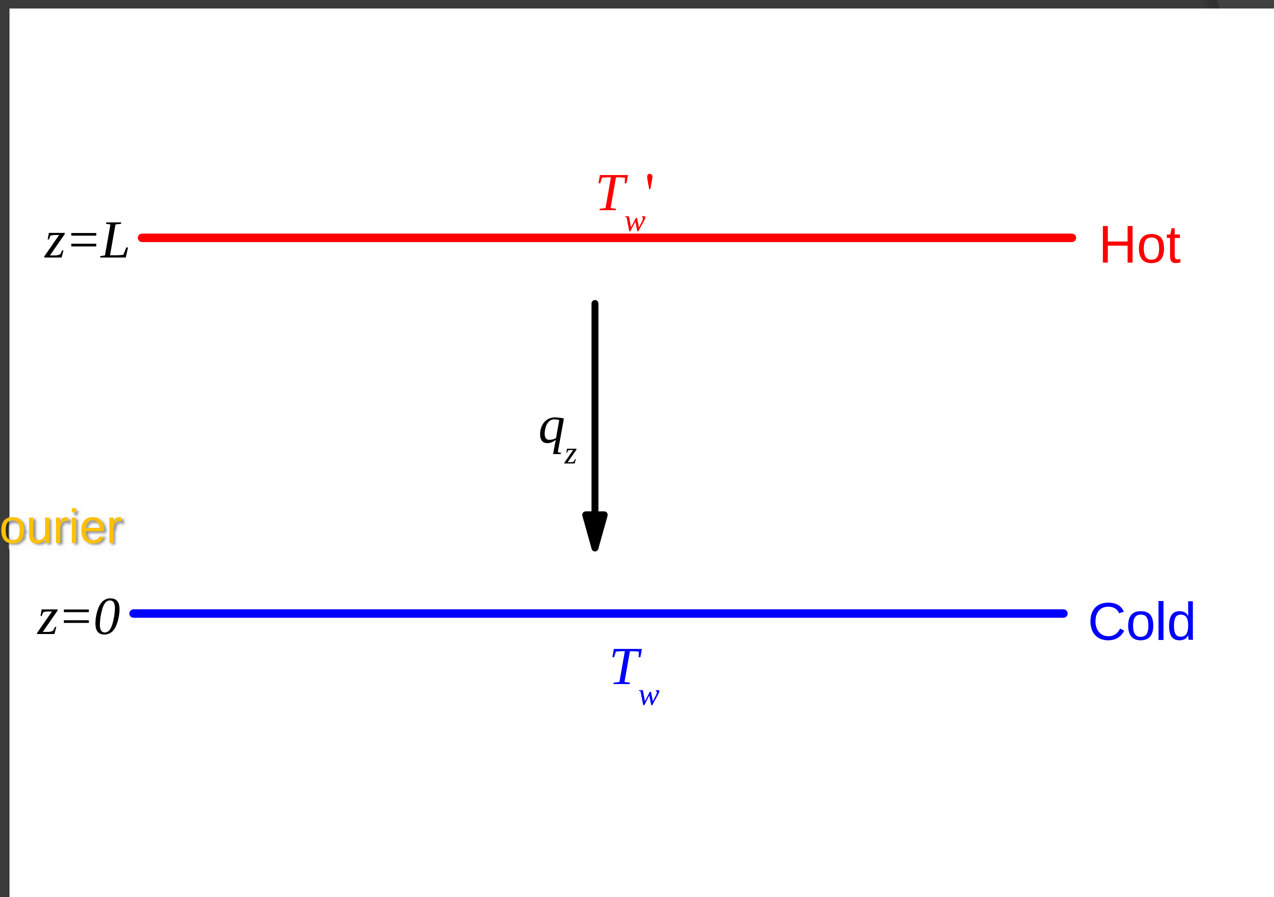
$$\mathbf{q} = -\kappa \nabla T, \qquad \boxed{\kappa = \frac{5}{2} \frac{n k_B^2 T}{m \lambda_{11}}}$$

George Gabriel Stokes
(1819-1903)

$$P_{ij} = p\delta_{ij} - \eta \left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3} \nabla \cdot \mathbf{u} \delta_{ij} \right), \qquad \boxed{\eta = \frac{p}{\lambda_{02}}}$$

1. Steady planar Fourier flow

Jean-Baptiste Joseph Fourier
(1768-1830)



Asmolov, Makashev, and Nosik (1979) proved that an exact solution of the (nonlinear) Boltzmann equation for Maxwell molecules exists with

$$p = n k_B T = \mathrm{const}$$

$$\mathbf{u} = \mathbf{0}$$

$$\frac{\partial}{\partial z} T \frac{\partial T}{\partial z} = 0 \Rightarrow T(z) = \sqrt{A + Bz}$$

Dimensionless quantities

(Local) Knudsen number:

$$\epsilon = \frac{\sqrt{2k_BT/m}}{\lambda_{11}}\frac{\partial \ln T}{\partial z}$$

Reduced distribution function:

$$\phi(\mathbf{c};\epsilon) = \frac{1}{n(z)}\left[\frac{2k_BT(z)}{m}\right]^{3/2} f(z,\mathbf{v}) \quad \text{“Normal” solution (Bulk)}$$

Reduced moments:

$$\mathcal{M}_{r\ell}(\epsilon) = \int d\mathbf{c}\,\Psi_{r\ell 0}(\mathbf{c})\phi(\mathbf{c};\epsilon)$$
$$M_{r\ell}(\epsilon) = \int d\mathbf{c}\,c^{2r}c_z^{\ell}\phi(\mathbf{c};\epsilon) = \text{Linear Combination}\{\mathcal{M}_{r'\ell'}, 2r'+\ell' \le 2r+\ell\}$$

Hierarchy of moment equations

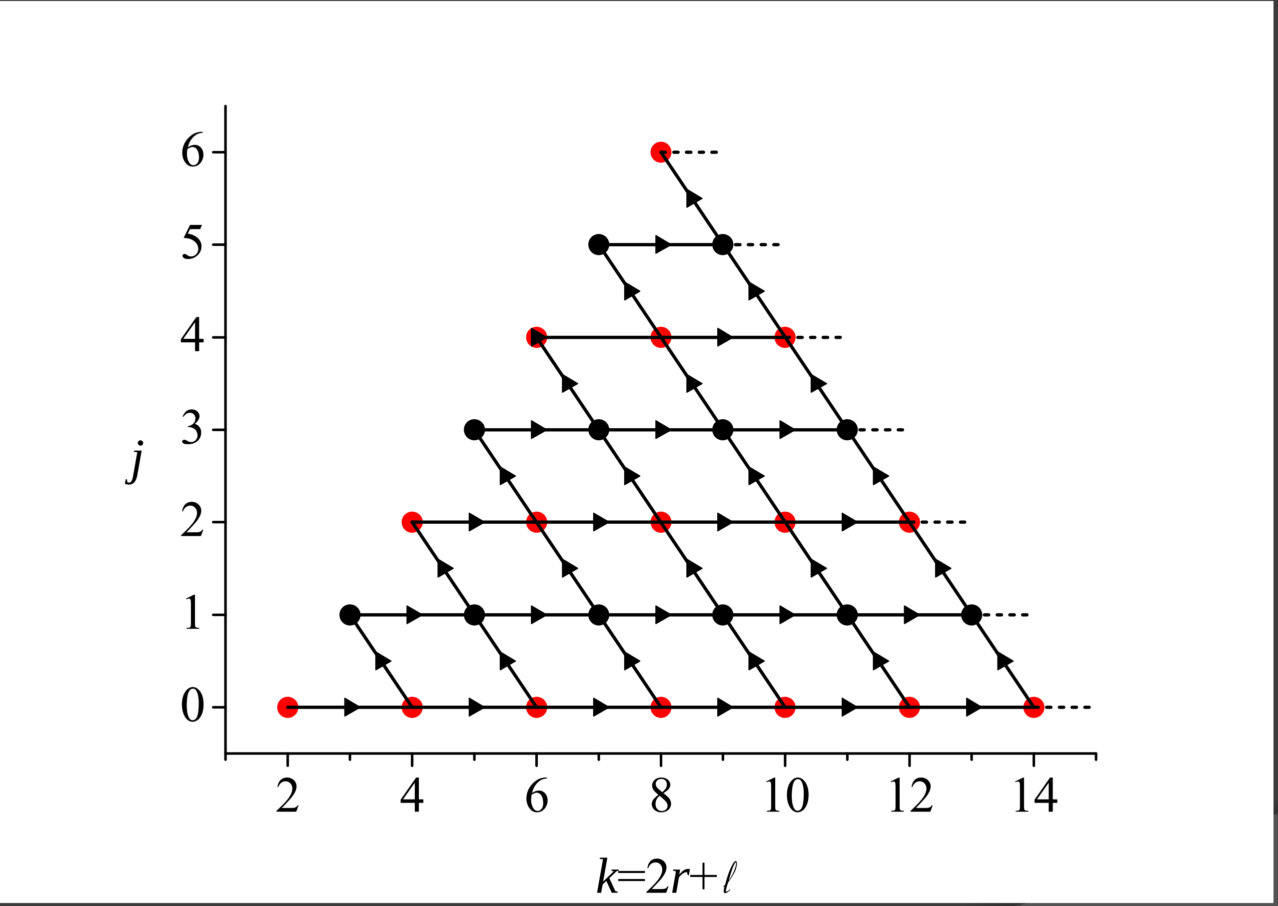
$$\frac{\epsilon}{2}\left(2r+\ell-1-\epsilon\frac{\partial}{\partial\epsilon}\right)M_{r,\ell+1}(\epsilon) \;=\; \frac{n}{\lambda_{11}}\int d\mathbf{c}\, c^{2r} c_z^{\ell} J[\mathbf{c}|\phi(\epsilon),\phi(\epsilon)]$$
$$\equiv\; J_{r\ell}(\epsilon)$$

The hierarchy admits a solution whereby the moments $M_{r\ell}(\epsilon)$ (with $2r+\ell\geq 2$) are *polynomials* in ϵ of degree $2r+\ell-2$ and parity ℓ:

$$M_{r\ell}(\epsilon)=\sum_{j=0}^{2r+\ell-2}\mu_j^{(r\ell)}\epsilon^j,\quad \mu_j^{(r\ell)}=0 \text{ if } j+\ell=\text{odd}$$

Both sides of the hierarchy are polynomials of degree $2r+\ell-2$. Equating the coefficients of both sides allows one to get $\mu_j^{(r\ell)}$ recursively

Sketch of the sequence followed in the recursive determination of the coefficients $\mu_j^{(r\ell)}$



First few moments

$$k = 2r + \ell = 2 : \begin{cases} M_{10} = \frac{3}{2} \\ M_{02} = \frac{1}{2} \Rightarrow P_{xx} = P_{yy} = P_{zz} = nk_B T \end{cases}$$

Fourier's law

$$k = 2r + \ell = 3 : \begin{cases} M_{11} = -\frac{5}{4}\epsilon \Rightarrow \boxed{q_z = -\frac{5}{2}\frac{nk_B^2 T}{m\lambda_{11}}\frac{\partial T}{\partial z}} \\ M_{03} = -\frac{3}{4}\epsilon \Rightarrow \langle v_z^3 \rangle = 3\langle v_x^2 v_z \rangle = \frac{3}{5}\langle v^2 v_z \rangle \end{cases}$$

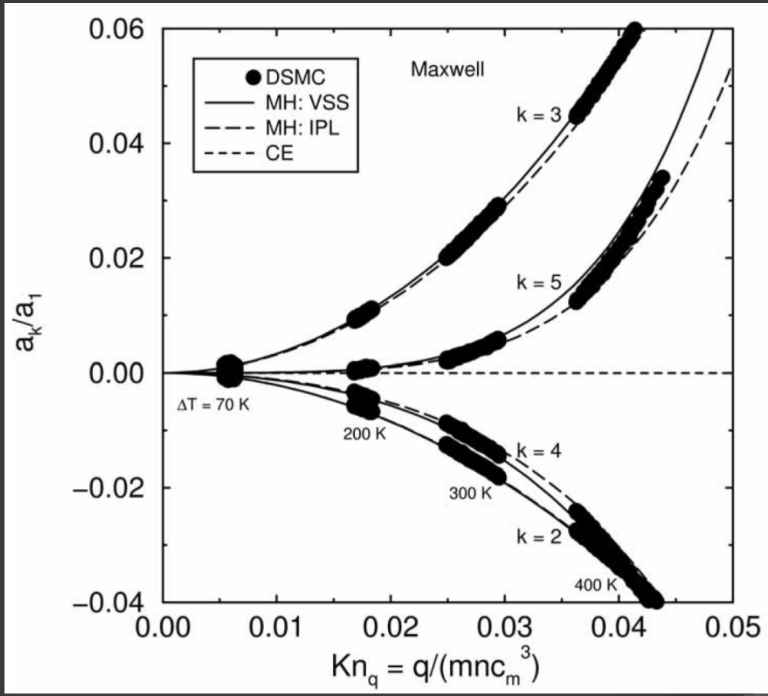
First few moments

$$k = 2r + \ell = 4 : \begin{cases} M_{20} = \frac{15}{4} + \frac{35}{4}\epsilon^2 \\ M_{12} = \frac{5}{4} + \frac{17}{4}\epsilon^2 \\ M_{04} = \frac{3}{4} + \frac{81}{28}\epsilon^2 \end{cases}$$

$$k = 2r + \ell = 5 : \begin{cases} M_{21} = -\frac{35}{4}\epsilon - \left(\frac{1163}{36} + \frac{7}{\lambda_{22}}\right)\epsilon^3 \\ M_{13} = -\frac{21}{4}\epsilon - \left[\frac{1163}{60} + \frac{21}{5\lambda_{22}} + \frac{27}{10\lambda_{13}}\left(\frac{8}{7} + \frac{1}{\lambda_{22}}\right)\right]\epsilon^3 \\ M_{05} = -\frac{15}{4}\epsilon - \frac{9}{2}\left[\frac{1163}{378} + \frac{2}{3\lambda_{22}} + \frac{2}{3\lambda_{13}}\left(\frac{8}{7} + \frac{1}{\lambda_{22}}\right)\right]\epsilon^3 \end{cases}$$

Benchmark for DSMC simulations

$$\frac{a_k(\epsilon)}{a_1(\epsilon)} \equiv \frac{15\sqrt{\pi}}{8}\sum_{r=1}^{k}\frac{(-1)^{r-1}k!}{r!(k-r)!(r+3/2)!}\frac{M_{r1}(\epsilon)}{M_{11}(\epsilon)}$$
$$= \text{Polynomial of degree } 2k-2$$



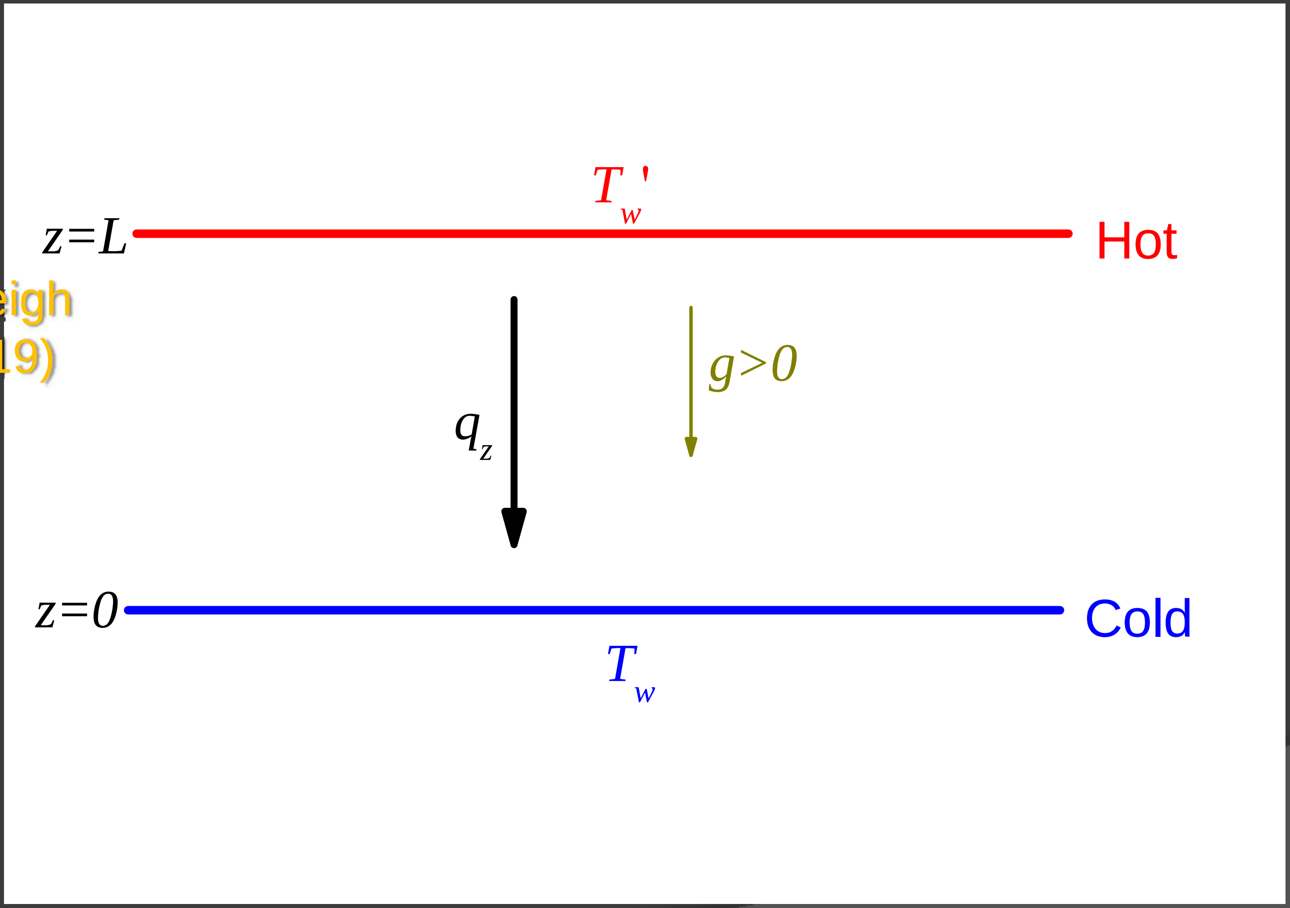
Gallis et al., Phys. Fluids **18**, 017104 (2006)

BGK description

- Same qualitative results for moments.
- Full velocity distribution function $\phi(\mathbf{c};\epsilon)$.
- Divergence of the (CE) expansion of $\phi(\mathbf{c};\epsilon)$ in powers of ϵ.

2. Steady planar Fourier flow with *gravity* (Rayleigh-Bénard-like flow)

Lord Rayleigh
(1842-1919)



Henri Bénard
(1874-1939)

Perturbation analysis

Tij, Garzó, and Santos, Phys. Rev. E **56**, 6729 (1997)

$$\underbrace{\gamma \equiv \frac{g}{\lambda_{11}^2}\frac{\partial \ln T}{\partial z}}_{\text{akin to the Rayleigh number}}, \qquad \mathrm{Ra} \sim |\gamma| \left(\frac{L}{\mathrm{m.f.p.}}\right)^4$$

$$|\gamma| \ll 1, \qquad M_{r\ell} = \underbrace{\boxed{M_{r\ell}^{(0)}}}_{\text{Pure Fourier flow}} + M_{r\ell}^{(1)}\gamma + M_{r\ell}^{(2)}\gamma^2 + \cdots$$

Corrections to Navier-Stokes

$$\gamma \equiv \frac{g}{\lambda_{11}^2}\frac{\partial \ln T}{\partial z}$$

$$\frac{\partial p}{\partial z} = -\rho g + \mathcal{O}(\gamma^3), \qquad \frac{P_{zz} - p}{p} = \frac{128}{45}\gamma^2 + \mathcal{O}(\gamma^3)$$

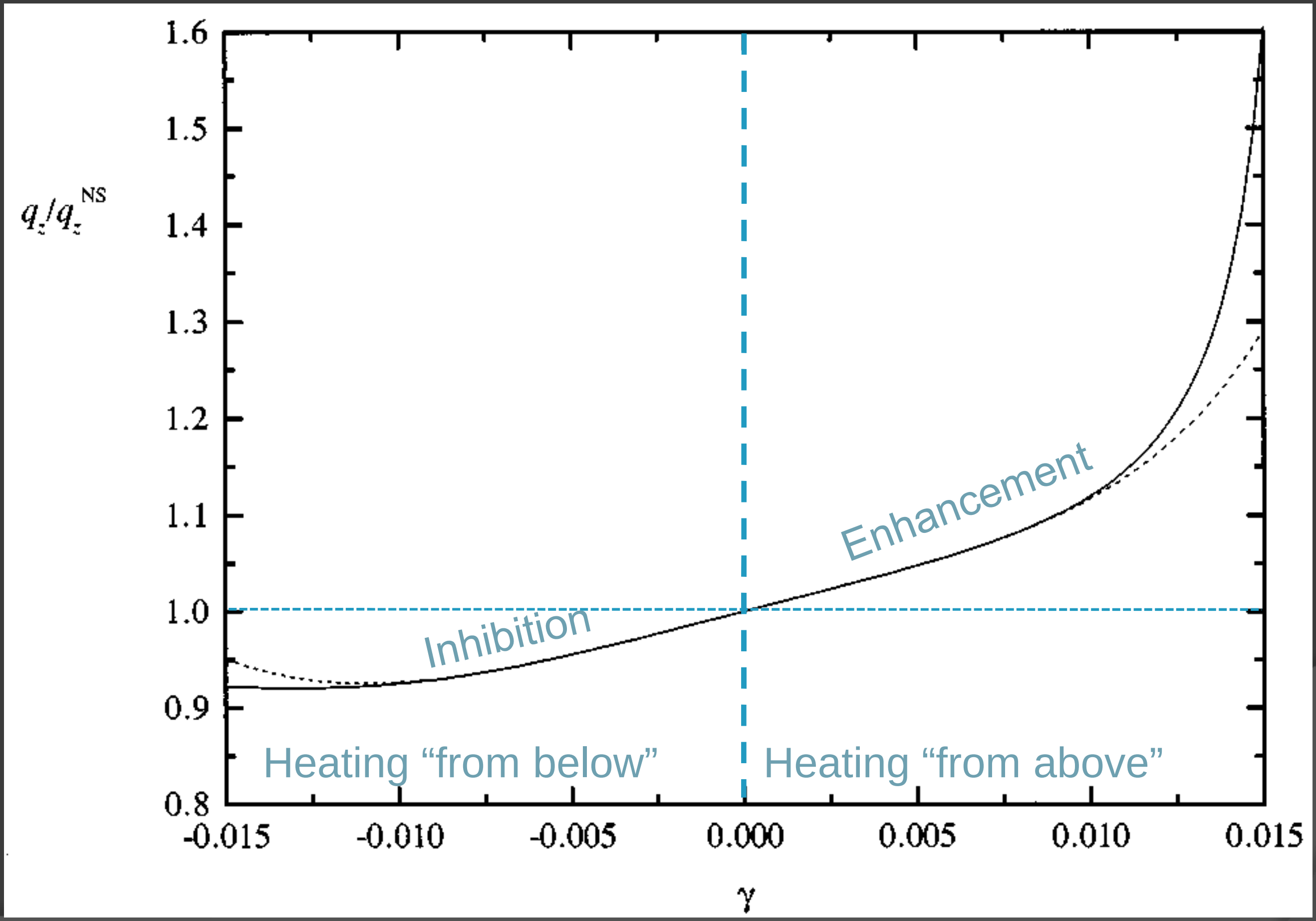
$$T\frac{\partial}{\partial z}T\frac{\partial T}{\partial z} = -\frac{104}{5}\frac{m(\lambda_{11}T)^2}{k_B}\gamma^2 + \mathcal{O}(\gamma^3)$$

$$q_z = q_z^{\mathrm{NS}}\left[1 + \frac{46}{5}\gamma + \mathcal{O}(\gamma^2)\right]$$

$$\frac{\langle v_z^3\rangle}{\langle v^2 v_z\rangle} = \frac{3}{5}\left[1 + \frac{64}{105}\gamma + \mathcal{O}(\gamma^2)\right]$$

Corrections to Navier-Stokes

$$\gamma \equiv \frac{g}{\lambda_{11}^2} \frac{\partial \ln T}{\partial z}$$

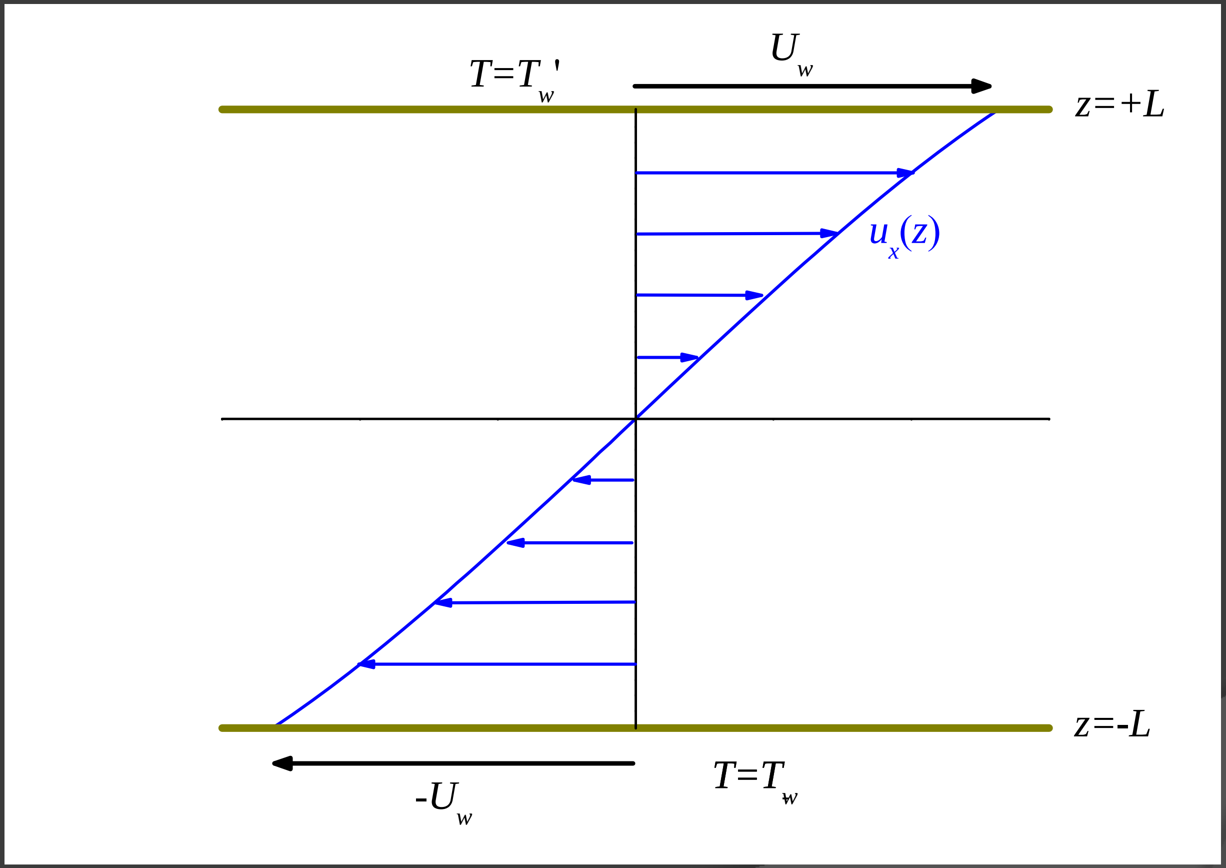


BGK description

- Same qualitative results.
- Moments up to order γ^6.
- Divergence of the expansion in powers of γ.

3. Steady planar Couette flow

Maurice Couette
(1858-1943)



Makashev and Nosik (1981) proved that an exact solution of the (nonlinear) Boltzmann equation for Maxwell molecules exists with

$$p = nk_B T = \text{const}$$

$$\frac{1}{\lambda_{02}} \frac{\partial u_x}{\partial z} = a = \text{const} \quad \text{(2nd Knudsen number)}$$

$$T \frac{\partial}{\partial z} T \frac{\partial T}{\partial z} = -\frac{3m(\lambda_{11} T)^2}{5k_B} a^2 \boxed{\theta(a)} = \text{const}$$

NS: $\theta(a)=1$

Dimensionless quantities

Knudsen numbers:

$$\epsilon = \frac{\sqrt{2k_BT/m}}{\lambda_{11}} \frac{\partial \ln T}{\partial z} \quad \text{(local)}, \qquad a = \frac{1}{\lambda_{02}} \frac{\partial u_x}{\partial z} \quad \text{(global)}$$

Reduced distribution function:

$$\phi(\mathbf{c}; \epsilon, a) = \frac{1}{n(z)} \left[\frac{2k_B T(z)}{m} \right]^{3/2} f(z, \mathbf{v}) \quad \text{“Normal” solution (Bulk)}$$

Reduced moments:

$$M_{r\ell h}(\epsilon, a) = \int d\mathbf{c}\, c^{2r} c_z^{\ell - h} c_x^h \phi(\mathbf{c}; \epsilon, a)$$

Hierarchy of moment equations

$$\left[\frac{\epsilon}{2}\left(2r+\ell-1-\epsilon\frac{\partial}{\partial\epsilon}\right)-\frac{6}{5}a^2\theta(a)\frac{\partial}{\partial\epsilon}\right]M_{r,\ell+1,h}(\epsilon)+\frac{3}{2}a\left(2rM_{r-1,\ell,h+1}+hM_{r,\ell,h-1}\right)$$

$$=\frac{n}{\lambda_{11}}\int d\mathbf{c}\,c^{2r}c_z^{\ell-h}c_x^h J[\mathbf{c}|\phi(\epsilon,a),\phi(\epsilon,a)]\equiv J_{r\ell h}(\epsilon,a)$$

The hierarchy admits a solution whereby the moments $M_{r\ell h}(\epsilon,a)$ (with $2r+\ell\geq 2$) are *polynomials* in ϵ of degree $2r+\ell-2$ and parity ℓ:

$$M_{r\ell h}(\epsilon,a)=\sum_{j=0}^{2r+\ell-2}\mu_j^{(r\ell h)}(a)\epsilon^j,\quad \mu_j^{(r\ell h)}(a)=0\text{ if }j+\ell=\text{odd}$$

Corrections to Navier-Stokes

$$a \equiv \frac{1}{\lambda_{02}} \frac{\partial u_x}{\partial z}$$

$$q_z = -\frac{5}{2} \frac{n k_B^2 T}{m \lambda_{11}} \kappa^*(a) \frac{\partial T}{\partial z}, \quad P_{xz} = -\frac{p}{\lambda_{02}} \eta^*(a) \frac{\partial u_x}{\partial z}, \quad \eta^*(a) = \theta(a) \kappa^*(a)$$

$$q_x = \frac{5}{2} \frac{n k_B^2 T}{m \lambda_{11}} \Phi(a) \frac{\partial T}{\partial z}$$

$$\frac{P_{xx} - P_{zz}}{p} = \Delta_1(a), \quad \frac{P_{yy} - P_{zz}}{p} = \Delta_2(a)$$

NS: $\kappa^* = \eta^* = \theta = 1, \quad \Phi = \Delta_1 = \Delta_2 = 0$

Perturbation analysis

Tij and Santos, Phys. Fluids **7**, 2857 (1995)

$$\kappa^*(a) = 1 - \left(\frac{1091}{150} - \frac{6\lambda_{04}}{1225}\right) a^2 + \mathcal{O}(a^4)$$

$$\eta^*(a) = 1 - \frac{149}{45} a^2 + \mathcal{O}(a^4)$$

$$\theta(a) = 1 + \left(\frac{1783}{450} - \frac{6\lambda_{04}}{1225}\right) a^2 + \mathcal{O}(a^4)$$

super-Burnett

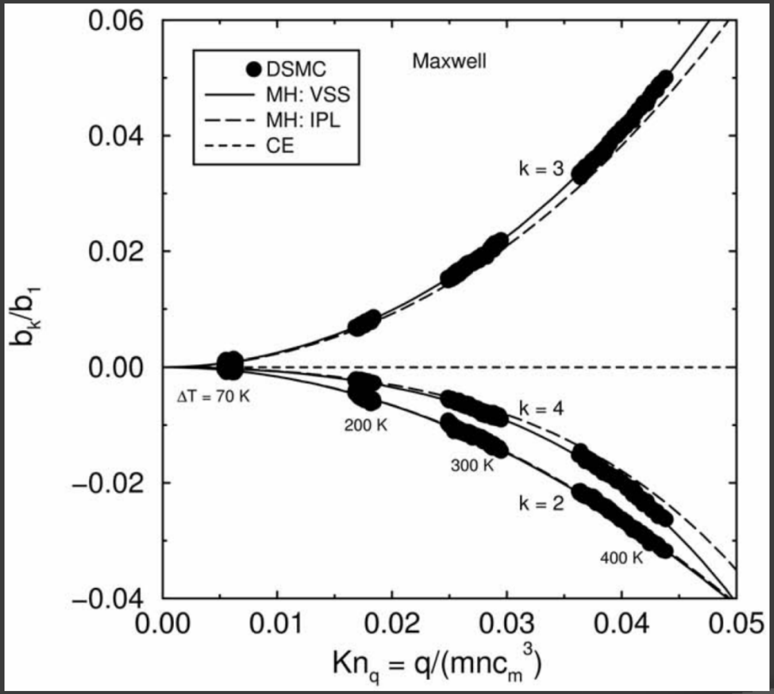
$$\Phi(a) = \frac{7}{2} a + \mathcal{O}(a^3)$$

$$\Delta_1(a) = \frac{14}{5} a^2 + \mathcal{O}(a^4), \quad \Delta_2(a) = \frac{4}{5} a^2 + \mathcal{O}(a^4)$$

Burnett

Benchmark for DSMC simulations

$$\frac{b_k(\epsilon)}{b_1(\epsilon)} \equiv \lim_{a\to 0} \frac{15\sqrt{\pi}}{8} \sum_{r=1}^{k} \frac{(-1)^{r-1}(k-1)!}{(r-1)!(k-r)!(r+3/2)!} \frac{M_{r-1,2,1}(\epsilon,a)}{M_{021}(\epsilon,a)}$$
$$= \text{Polynomial of degree } 2k-2$$



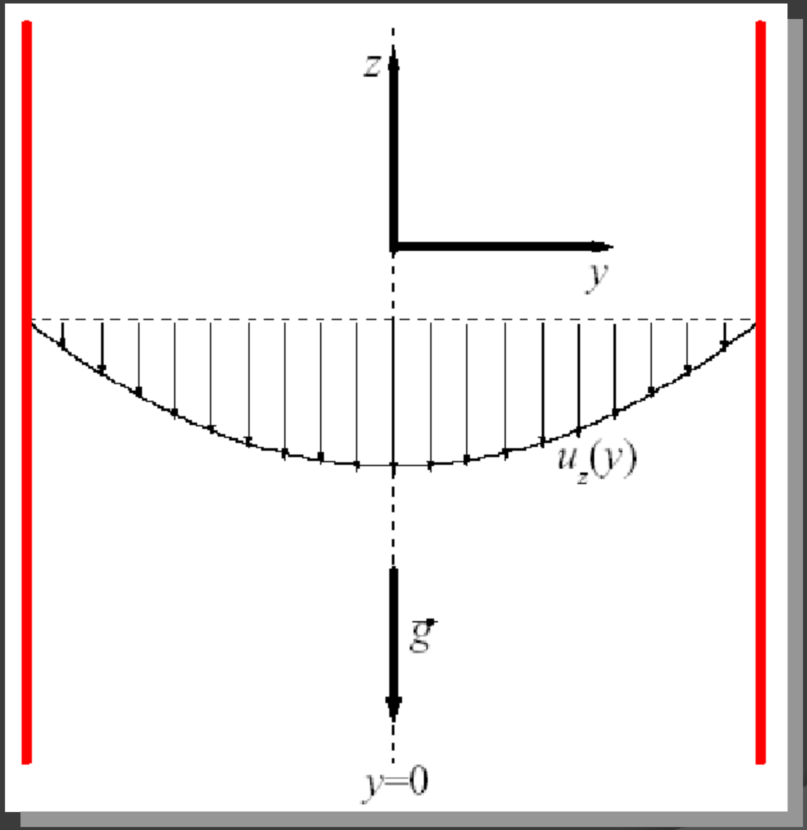
Gallis et al., Phys. Fluids **18**, 017104 (2006)

BGK description

- Same qualitative results for moments.
- Explicit expressions for $\kappa^*(a)$, $\eta^*(a)$, $\theta(a)$, $\Phi(a)$, $\Delta_1(a)$, and $\Delta_2(a)$.
- Full velocity distribution function $\phi(\mathbf{c};\epsilon,a)$.
- Divergence of the expansion in powers of a.
- Influence of gravity analyzed.

4. Force-driven Poiseuille flow

Jean-Louis-Marie Poiseuille
(1797-1869)



NAVIER-STOKES (NEWTONIAN) DESCRIPTION

$$P_{xx} = P_{yy} = P_{zz} = p$$

Equal normal stresses

$$P_{yz} = -\eta \frac{\partial u_z}{\partial y}$$

Newton's law

$$q_y = -\kappa \frac{\partial T}{\partial y}$$

Fourier's law

$$q_z = 0$$

No longitudinal heat flux

$$p(y) = p_0 = \text{const}$$

$$u_z(y) = u_0 + \frac{\rho_0 g}{2\eta_0} y^2 + \mathcal{O}(g^3)$$

$$T(y) = T_0 - \frac{\rho_0^2 g^2}{12\eta_0 \kappa_0} y^4 + \mathcal{O}(g^4)$$

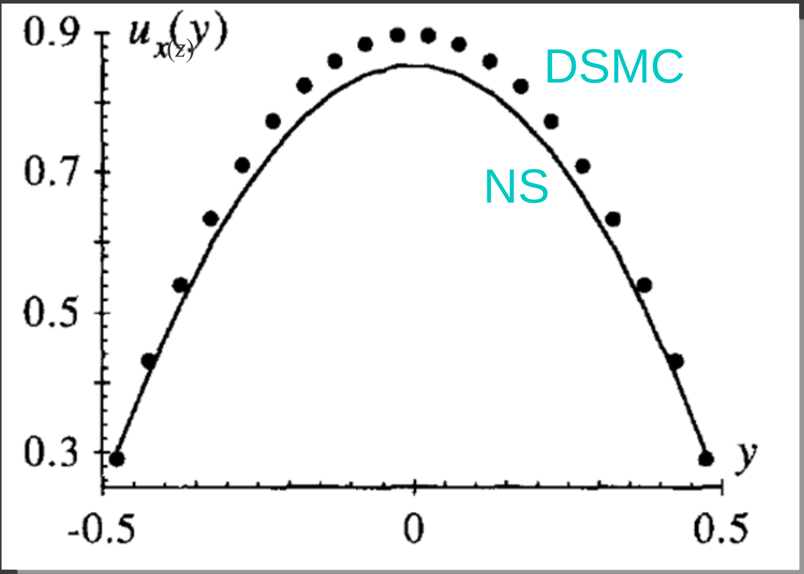
Temperature is *maximal* at the central layer ($y=0$)

Do NS predictions agree with computer simulations?

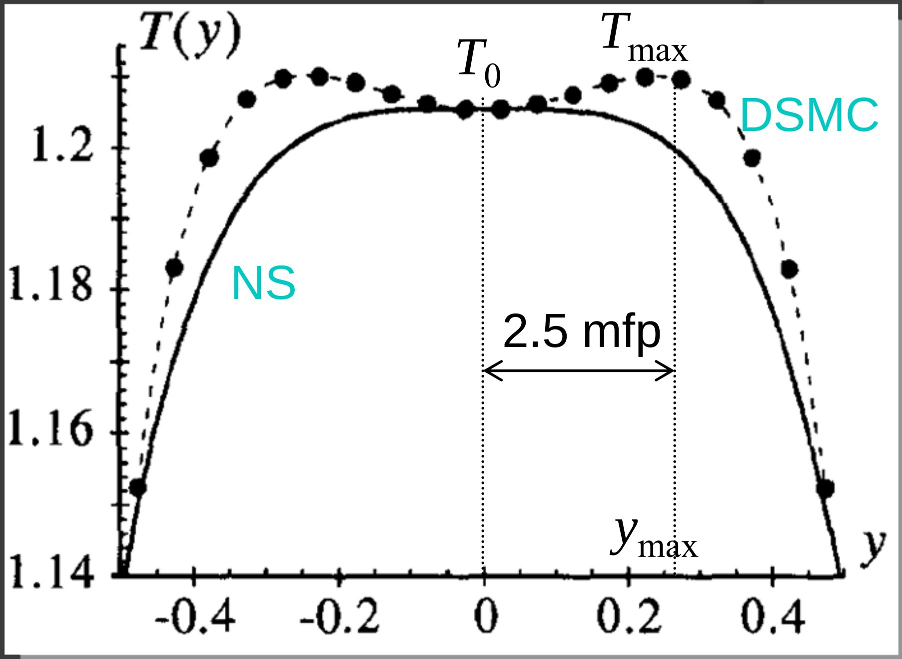
On the validity of hydrodynamics in plane Poiseuille flows

M. Malek Mansour[a,*], F. Baras[a], Alejandro L. Garcia[b,1]

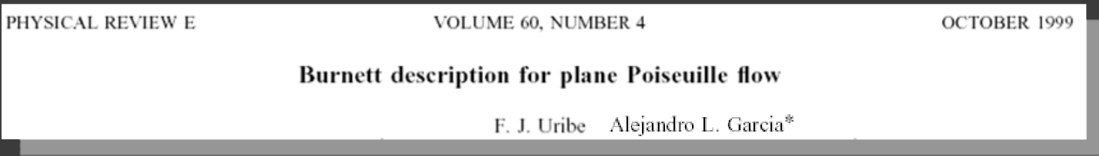
Physica A 240 (1997) 255–267



but ...



A Burnett-order effect?


PHYSICAL REVIEW E VOLUME 60, NUMBER 4 OCTOBER 1999

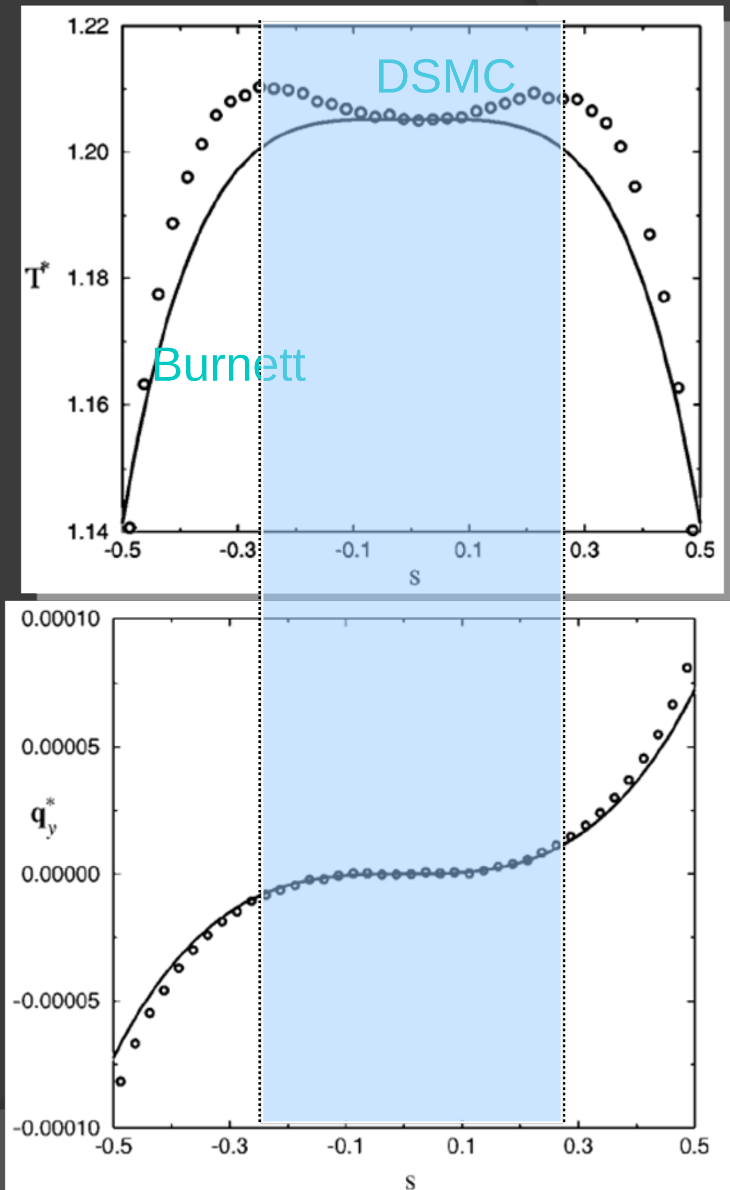
Burnett description for plane Poiseuille flow

F. J. Uribe Alejandro L. Garcia*

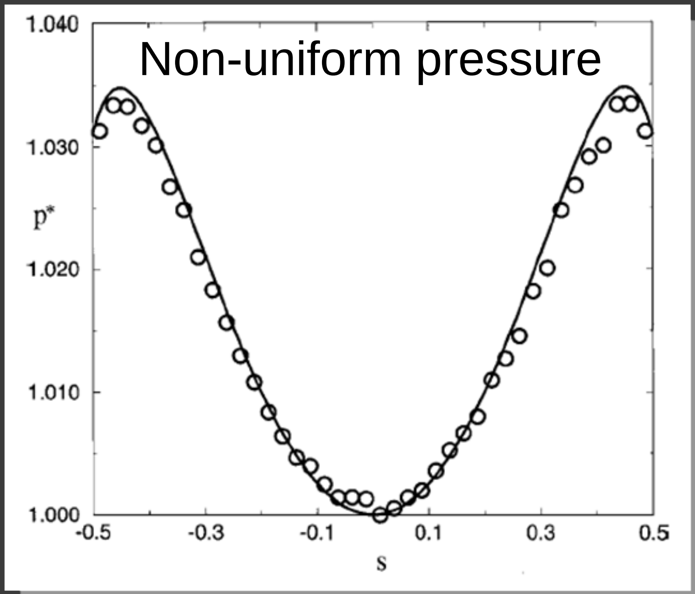
In the slab $y<|y_{\max}|$,

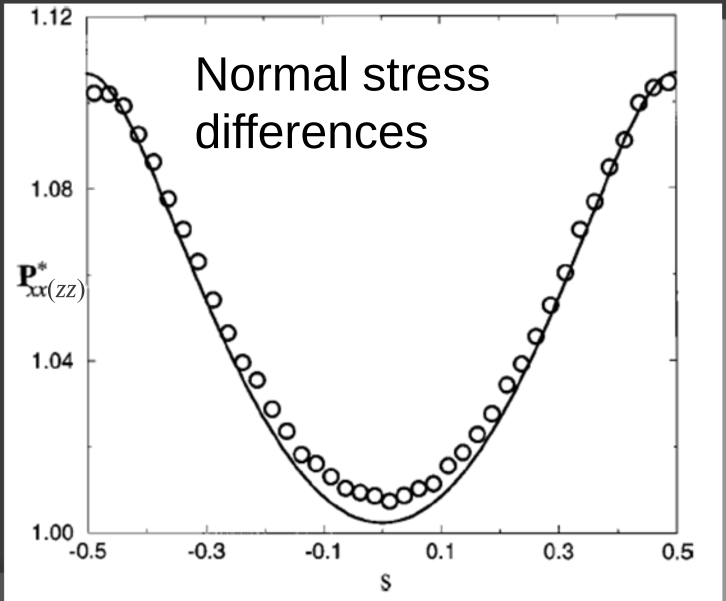
$$\boxed{\operatorname{sgn} q_y = \operatorname{sgn} \partial T/\partial y}$$

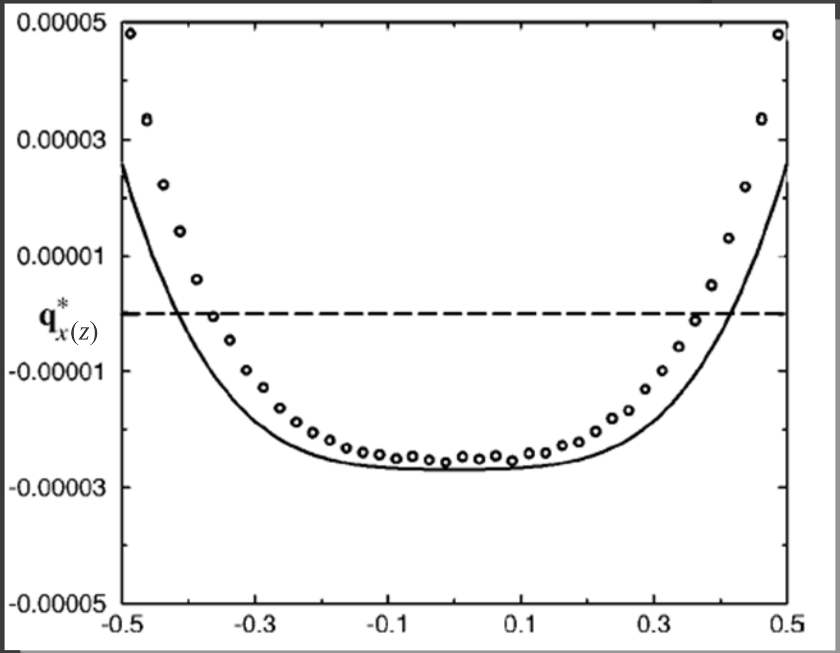
Heat flows from the colder to the hotter layers!!



Other Non-Newtonian properties







Longitudinal component of the heat flux
(but no longitudinal thermal gradient!)

Hierarchy of moment equations

$$M_{k_1,k_2,k_3}(y;g) = \int d\mathbf{v}\, v_x^{k_1} v_y^{k_2} [v_z - u_z(y,g)]^{k_3} f(y,\mathbf{v};g)$$

$$\frac{\partial}{\partial y} M_{k_1,k_2+1,k_3} + k_3 \left(\frac{\partial u_z}{\partial y} M_{k_1,k_2+1,k_3-1} + g M_{k_1,k_2,k_3-1} \right) = J_{k_1,k_2,k_3}$$

Perturbation analysis

Tij, Sabanne, and Santos, Phys. Fluids **10**, 1021 (1998)

$$M_{\vec{k}}(y;g) = \underbrace{M_{\vec{k}}^{(0)}}_{\text{Equilibrium moments (at } y=0)} + \sum_{j=1}^{\infty} M_{\vec{k}}^{(j)}(y) g^j$$

$$M_{\vec{k}}^{(1)}(y) = \text{Linear function of } y$$

$$M_{\vec{k}}^{(j)}(y) = \text{Polynomial in } y \text{ of degree } 2j, \quad j \geq 2$$

Results to order g^2: Hydrodynamic fields

$$u_z(y) = u_0 + \frac{\rho_0 g}{2\eta_0} y^2 + \mathcal{O}(g^3)$$

$$p(y) = p_0 \left[1 + C_p \left(\frac{mg}{T_0} \right)^2 y^2 \right] + \mathcal{O}(g^4)$$

$$T(y) = T_0 \left[1 - \frac{\rho_0^2 g^2}{12\eta_0 \kappa_0 T_0} y^4 + C_T \left(\frac{mg}{T_0} \right)^2 y^2 \right] + \mathcal{O}(g^4)$$

Results to order g^2: Hydrodynamic fluxes

$$P_{zz}(y) = p_0\left[1+\frac{7}{3}C_p\left(\frac{mg}{T_0}\right)^2 y^2 + C_{zz}\frac{\rho_0\eta_0^2 g^2}{p_0^3}\right] + \mathcal{O}(g^4)$$

$$P_{yy} = p_0\left(1-C_{yy}\frac{\rho_0\eta_0^2 g^2}{p_0^3}\right) + \mathcal{O}(g^4)$$

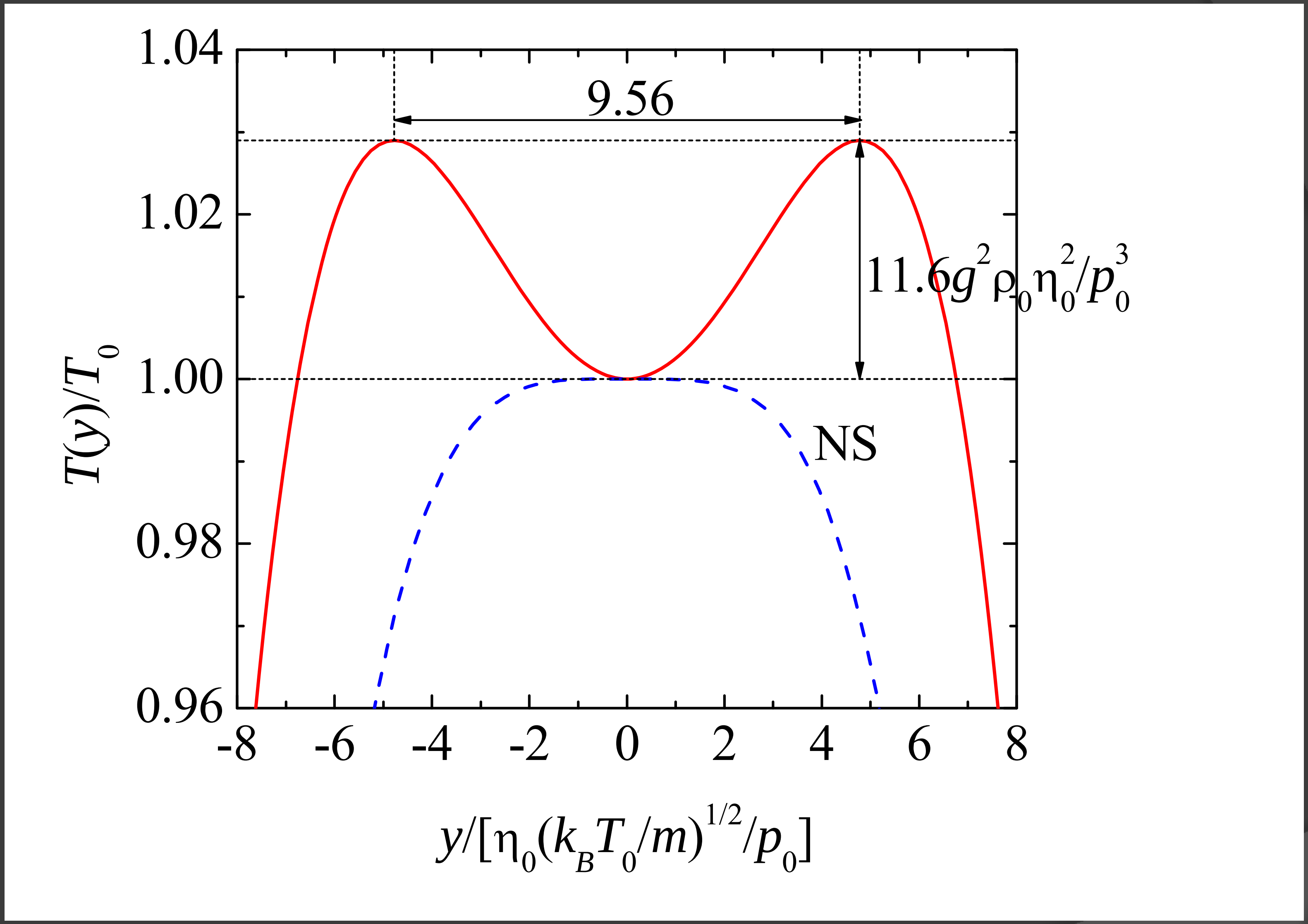
$$P_{yz}(y) = -\rho_0 g y\left[1 + \frac{\rho_0^2 g^2}{60\eta_0\kappa_0 T_0}y^4 + \frac{C_p - C_T}{3}\left(\frac{mg}{T_0}\right)^2 y^2\right] + \mathcal{O}(g^5)$$

$$q_y(y) = \frac{\rho_0^2 g^2}{3\eta_0}y^3 + \mathcal{O}(g^4), \quad q_z(y) = C_q m g \kappa_0 + \mathcal{O}(g^3)$$

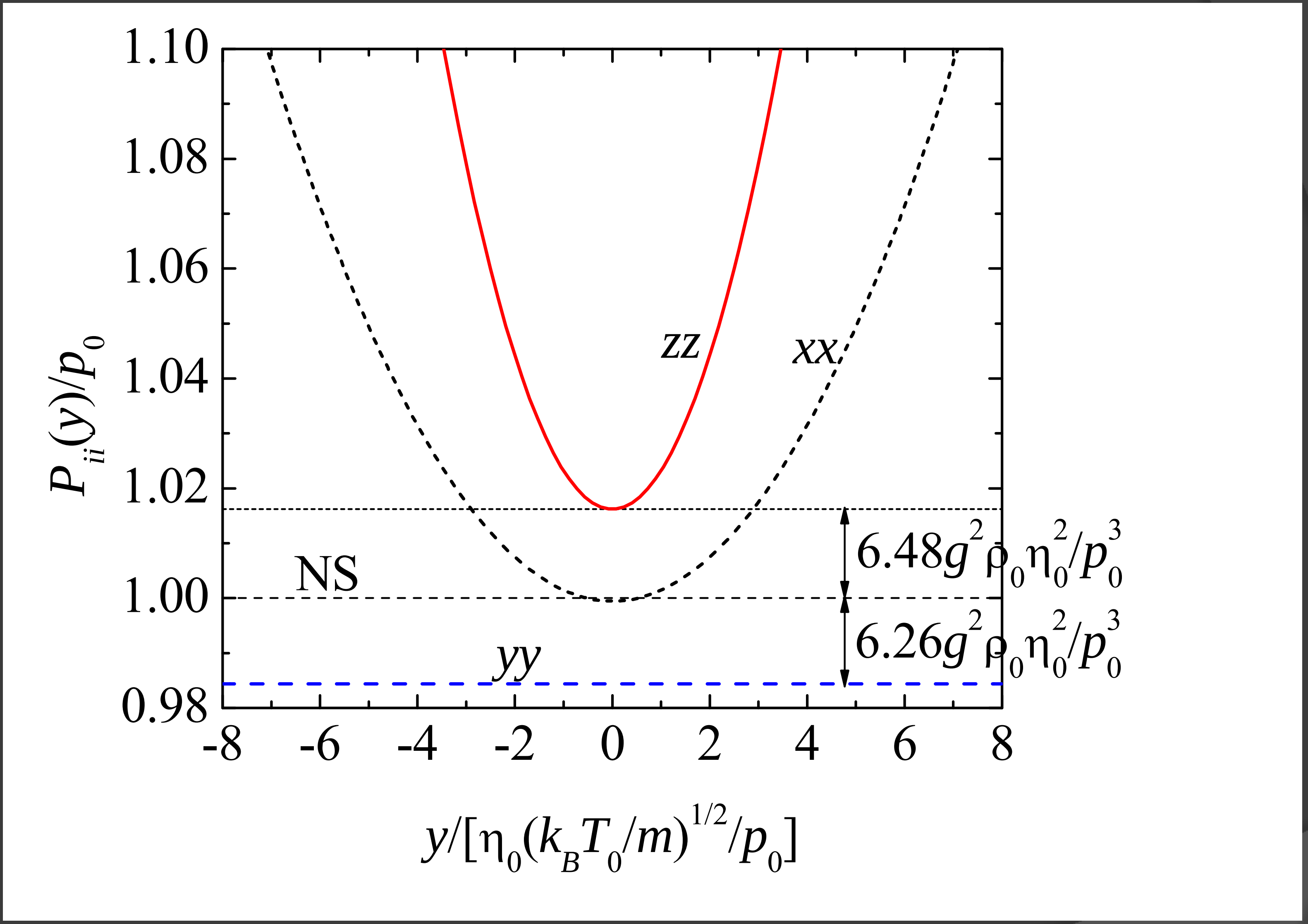
Numerical values of the coefficients

Coefficient	Quantity	NS	Burnett[a]	13-moment[b]	R13-moment[c]	19-moment[d]	Exact
C_p	p	0	1.2	1.2	1.2	1.2	1.2
C_T	T	0	0	0.56	0.9295	1.04	1.0153
C_{zz}	P_{zz}	0	0	0	3.413	?	6.4777
C_{yy}	P_{yy}	0	0	0	3.36	?	6.2602
C_q	q_z	0	0.4	0.4	0.4	0.4	0.4

[a]Uribe & Garcia (1999)
[b]Risso & Cordero (1998)
[c]Taheri, Struchtrup & Torrilhon (2008)
[d]Hess & Malek Mansour (1999)



$$g = 0.05 p_0^{3/2} / \rho_0^{1/2} \eta_0$$



$$g = 0.05 p_0^{3/2} / \rho_0^{1/2} \eta_0$$

BGK description

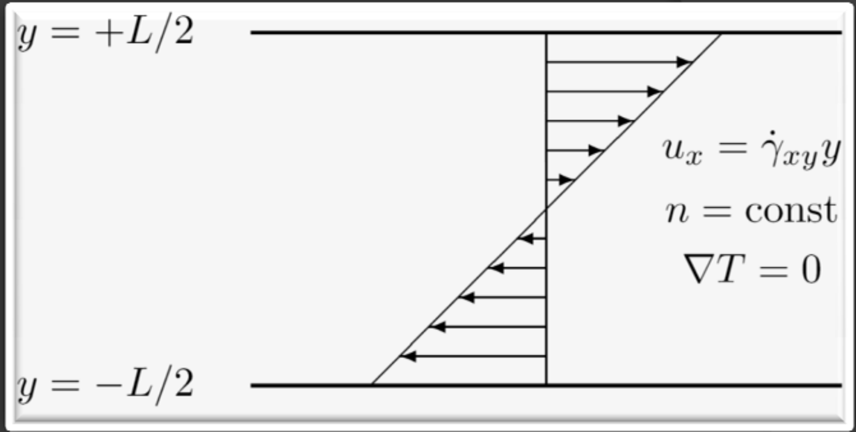
- Same qualitative results.
- Hydrodynamic fields up to order g^5.
- Velocity distribution function up to order g^3.
- Divergence of the expansion in powers of g.
- Exact non-perturbative solution for a special value of g.

5. Other (quasi-uniform) states

Uniform Shear Flow

$$n(t) = n(0)$$

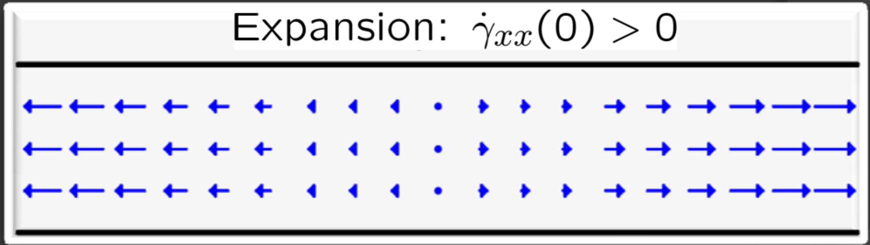
$$\dot{\gamma}_{xy}(t) = \dot{\gamma}_{xy}(0)$$

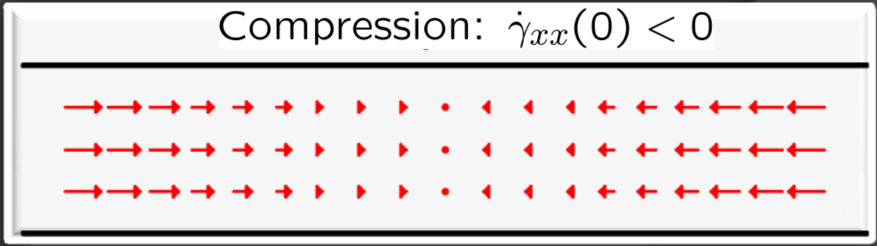


Uniform Longitudinal Flow

$$n(t) = \frac{n(0)}{1 + \dot{\gamma}_{xx}(0)t}$$

$$\dot{\gamma}_{xx}(t) = \frac{\dot{\gamma}_{xx}(0)}{1 + \dot{\gamma}_{xx}(0)t}$$





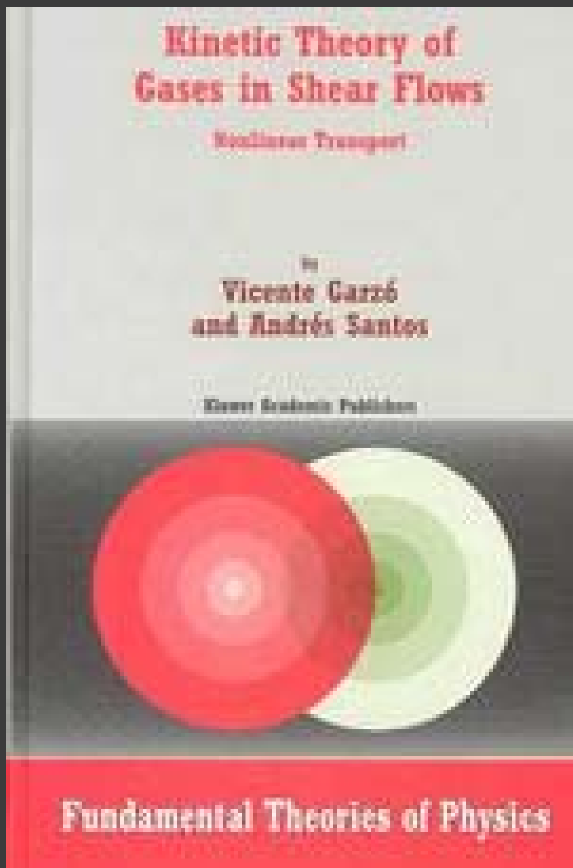
In both cases,

- Exact rheological functions for *arbitrary* values (*non-perturbative* solution) of the corresponding Knudsen number ($a=\gamma_{xy}/\lambda_{02}$, $a=\gamma_{xx}/\lambda_{02}$).
- Divergence of the fourth-degree moments beyond a *critical* value $a=a_c$.
- Algebraic high-velocity decay of the velocity distribution function for any a: Finite number of convergent moments.

Conclusions

- The hierarchy of moment equations can be recursively solved in the case of Maxwell molecules for some non-trivial states.
- No need to apply truncation closures.
- In some cases, a perturbation analysis is required.
- Exact results are important by themselves and also useful as an assessment of
  - simulation techniques
  - approximate methods
  - kinetic models

More details about shear flows (including mixtures, BGK model description, …)



(Springer, 2003)

Acknowlegments (in alphabetical order):

- J. Javier Brey, Universidad de Sevilla, Spain
- James. W. Dufty, University of Florida, USA
- Vicente Garzó, Universidad de Extremadura, Spain
- Mohamed Tij, Université Moulay Ismaïl, Morocco

THANKS FOR YOUR ATTENTION!