THE SECOND AND THIRD SONINE COEFFICIENTS OF A FREELY COOLING GRANULAR GAS REVISITED

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Outline

- The freely cooling granular gas. Sonine coefficients.
- Brief review of previous results.
- Linear approximations. Comparison with DSMC results.
- Conclusions.

Basic state of a granular fluid: Freely cooling or Homogeneous Cooling State (HCS)

This state is:

✓Homogeneous.

✓Isotropic.

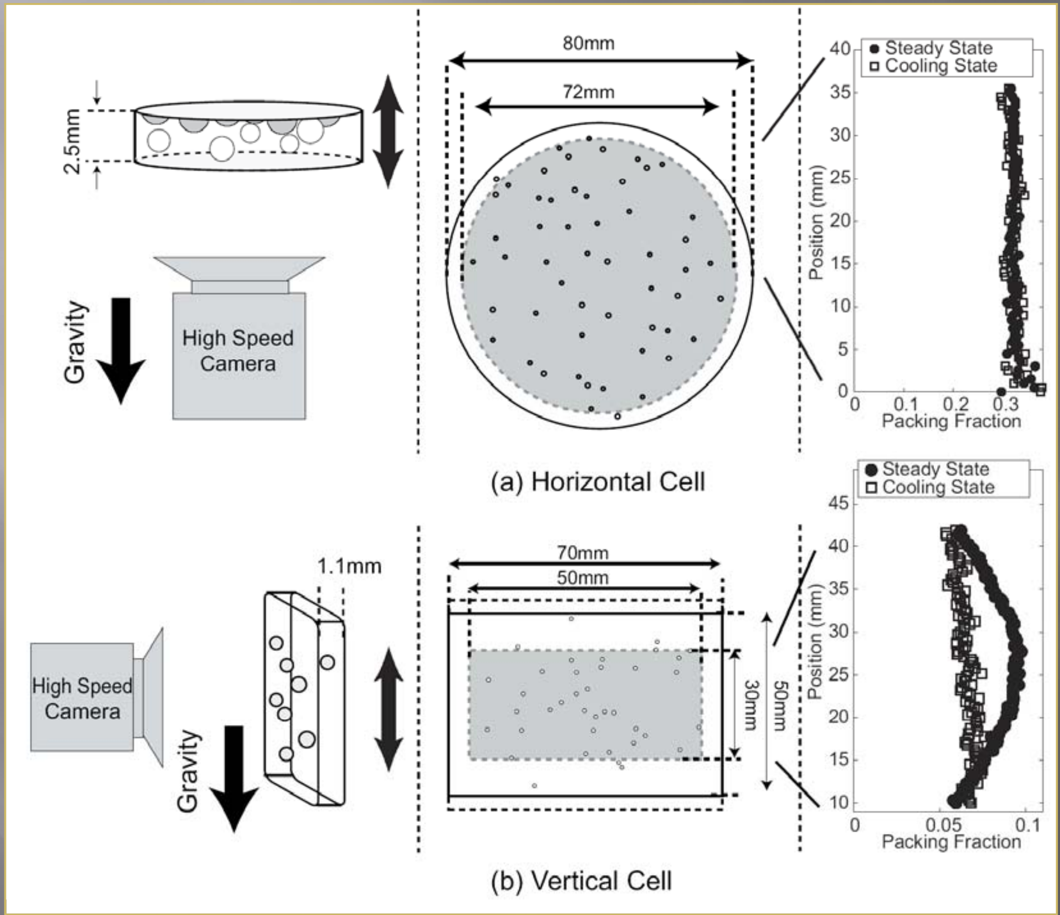
✓The granular temperature monotonically decreases in time (Haff's law).

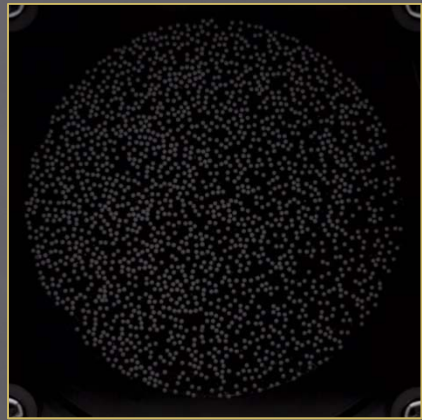
✓... But the distribution function of the rescaled velocities reaches a stationary form (self-similarity solution).

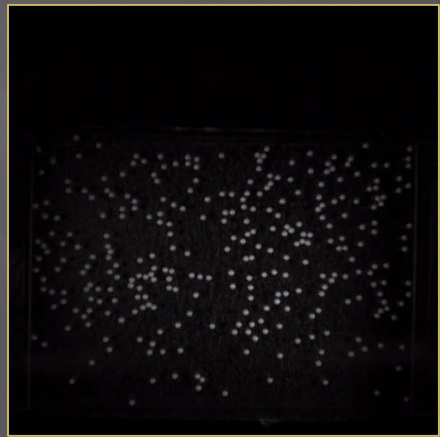
✓And it is unstable! (avoid long times and/or large dimensions).

Free cooling under microgravity
(Tatsumi, Murayama, Hayakawa, and Sano, unpublished)

(Videoclips courtesy of S. Tatsumi)



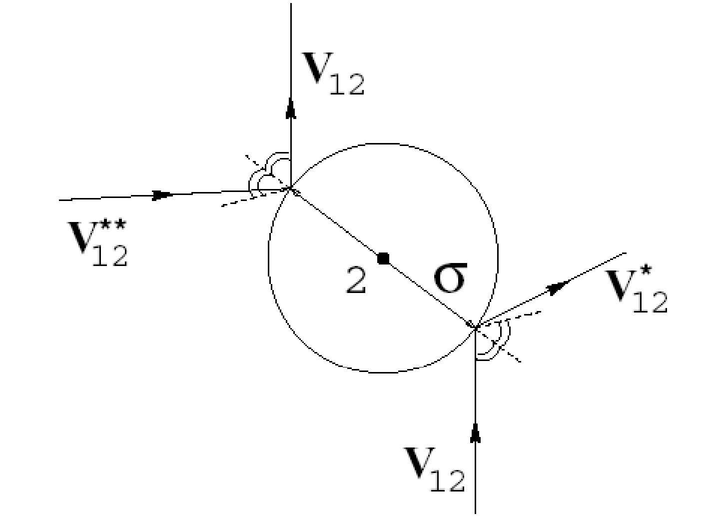




Minimal model of a granular gas: A gas of (smooth) *inelastic* hard spheres

Several circles (Kandinsky, 1926)

Minimal model of a granular gas: A gas of (smooth) *inelastic* hard spheres



(After T.P.C. van Noije & M.H. Ernst)

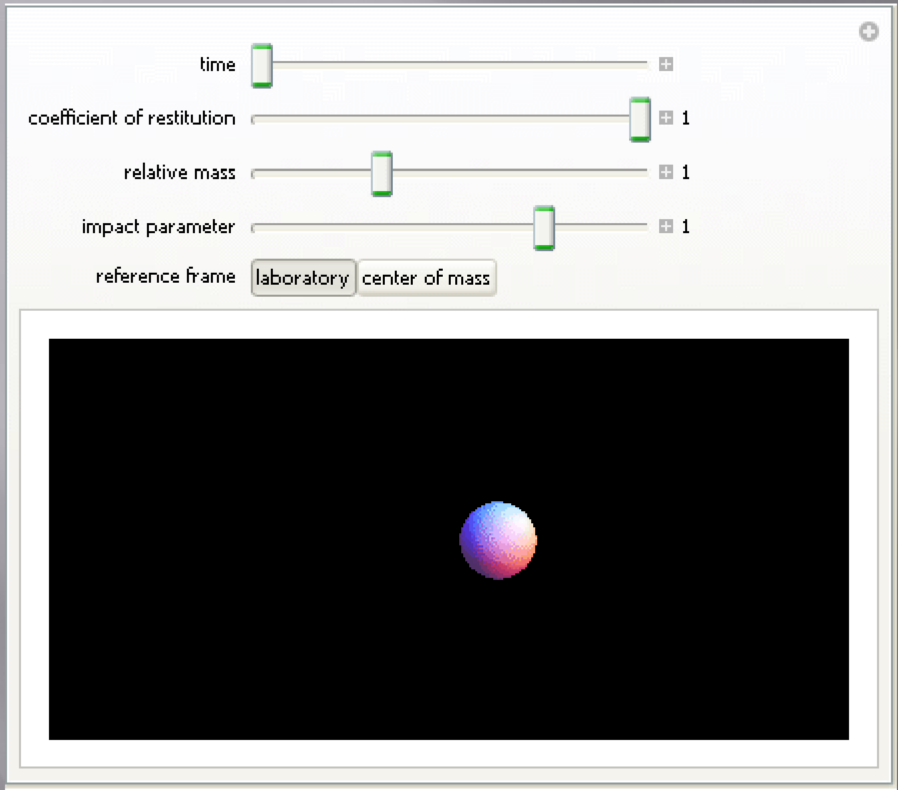
- Mass m
- Diameter σ
- Coefficient of normal restitution α
- $\alpha=1$ for elastic collisions

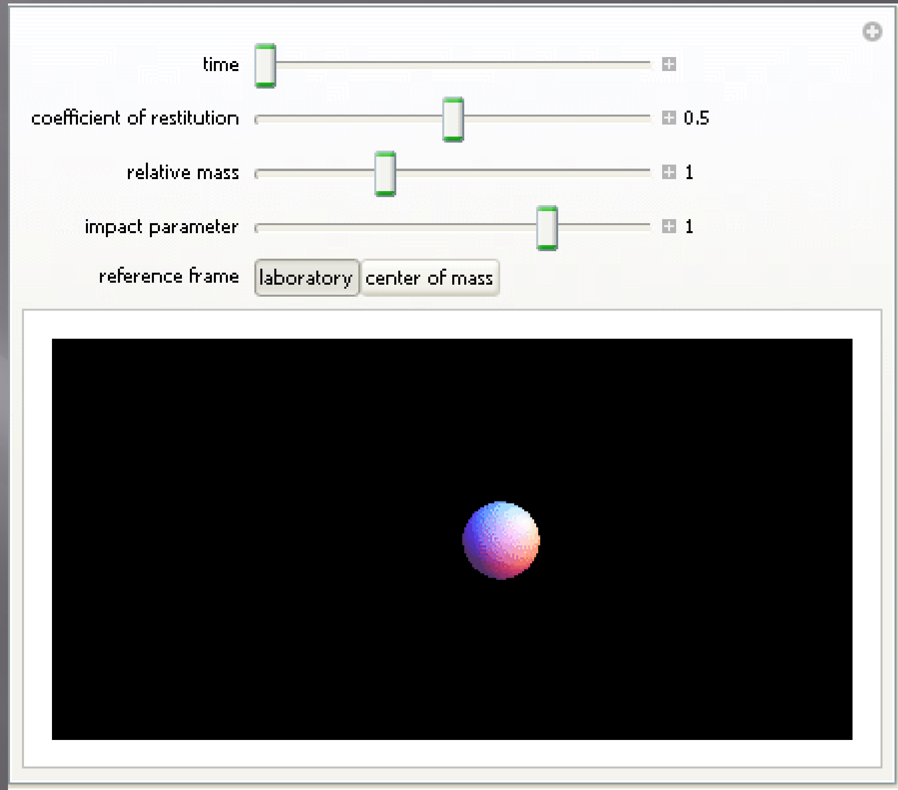
Relative velocity

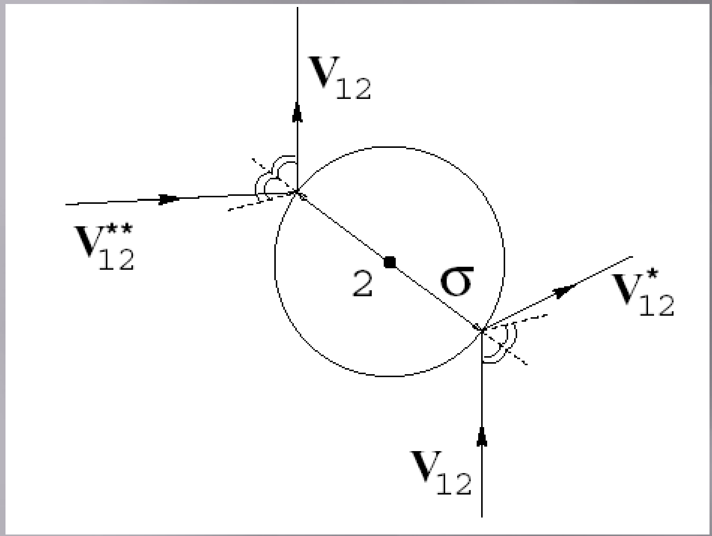
Direct collision: $\mathbf{v}_1^* = \mathbf{v}_1 - \frac{1+\alpha}{2}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2^* = \mathbf{v}_2 + \frac{1+\alpha}{2}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}$

Restituting collision: $\mathbf{v}_1^{**} = \mathbf{v}_1 - \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2^{**} = \mathbf{v}_2 + \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}$

http://demonstrations.wolfram.com/InelasticCollisionsOfTwoSpheres/







Collisions conserve momentum, but not kinetic energy:

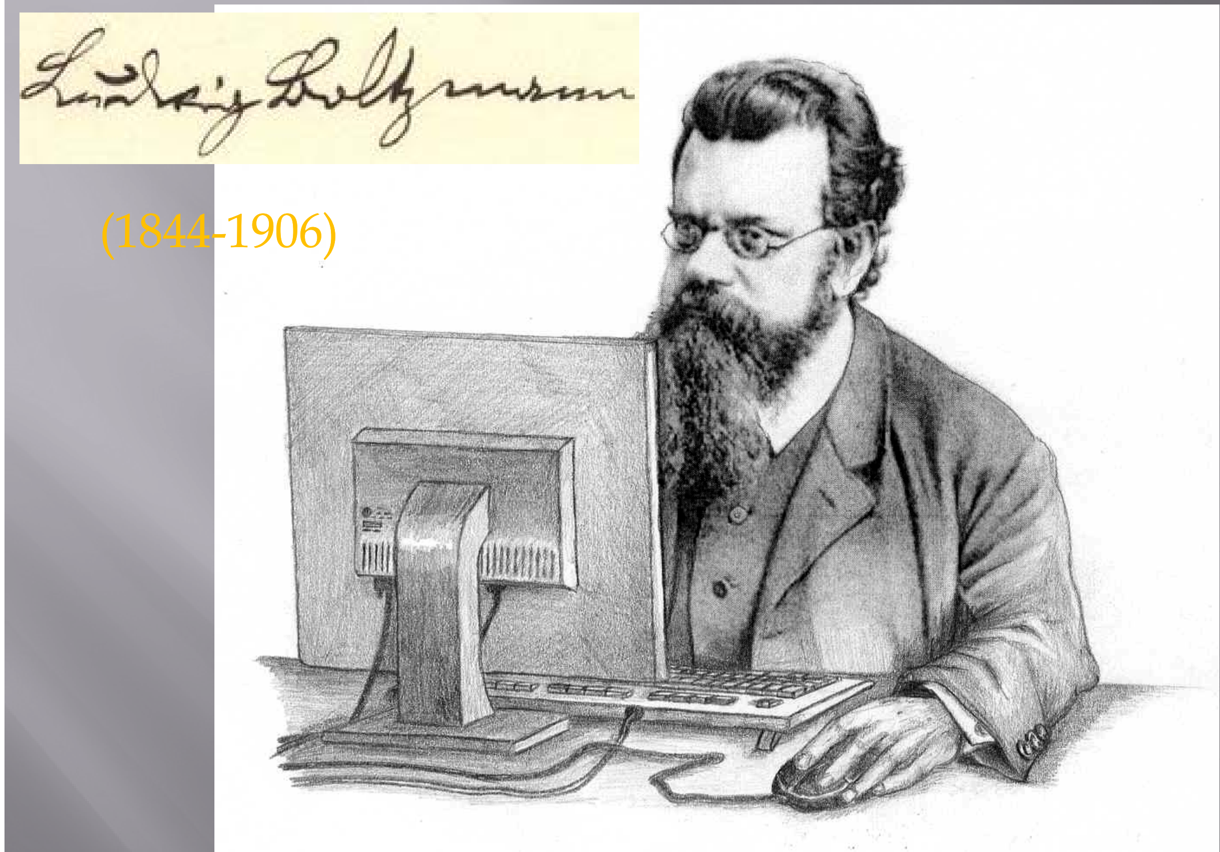
$$\Delta E = \frac{1}{2}m(v_1^{*2} + v_2^{*2} - v_1^2 - v_2^2) = -\frac{m}{2}(1-\alpha^2)(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})^2$$

"Granular" temperature: $T = \frac{m}{d}\langle(\mathbf{v}-\mathbf{u})^2\rangle, \quad \mathbf{u} = \langle\mathbf{v}\rangle$

$$\left.\frac{\partial T}{\partial t}\right|_{\mathrm{coll}} = -\zeta T, \quad \zeta \sim 1-\alpha^2$$

"Cooling" rate

The Enskog-Boltzmann equation (molecular chaos)



(1844-1906)

(Cartoon by Bernhard Reischl, University of Vienna)

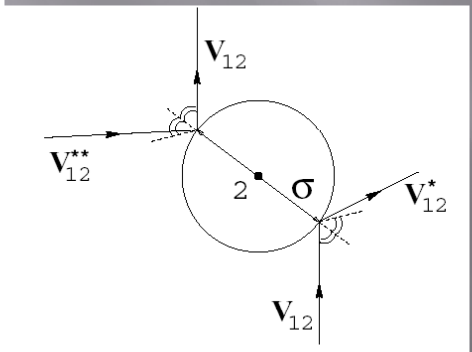
David Enskog
(1884-1947)

Enskog-Boltzmann equation (HCS)

$$\partial_t f(\mathbf{v}_1,t) = \boxed{J[\mathbf{v}_1|f(t),f(t)]}$$

Collision operator

$$J[\mathbf{v}_1|f(t),f(t)] = \chi\sigma^{d-1}\int \mathrm{d}\mathbf{v}_2 \int \mathrm{d}\widehat{\boldsymbol{\sigma}}\,\Theta(\mathbf{v}_{12}\cdot\widehat{\boldsymbol{\sigma}})(\mathbf{v}_{12}\cdot\widehat{\boldsymbol{\sigma}}) \times\left[\alpha^{-2} f(\mathbf{v}_1^{**},t)f(\mathbf{v}_2^{**},t) - f(\mathbf{v}_1,t)f(\mathbf{v}_2,t)\right]$$



$$\mathbf{v}_1^{**} = \mathbf{v}_1 - \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2^{**} = \mathbf{v}_2 + \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}$$

HCS

Thermal speed: $v_0(t) \equiv \sqrt{2T(t)/m} \equiv \sqrt{\frac{2}{d}\langle v^2\rangle_t}$

$= \frac{v_0(0)}{1+\zeta(0)t/2}$ Haff's law

Scaled distribution: $f(\mathbf{v},t) = n v_0^{-d}(t) F(c), \quad \mathbf{c}(t) = \frac{\mathbf{v}}{v_0(t)}$

High-velocity tail: $F(c) \sim e^{-Ac}$

Sonine coefficients

Thermal velocities: $F(c) = \pi^{-d/2} e^{-c^2}\left[1 + \sum_{k=2}^{\infty} a_k L_k^{(\frac{d-2}{2})}(c^2)\right]$

$$\langle c^4\rangle = \frac{d(d+2)}{4}(1+a_2)\,, \quad \langle c^6\rangle = \frac{d(d+2)(d+4)}{8}(1+3a_2-a_3)$$

Accurate determination of a_2 (and a_3) is important to characterize the deviation of $F(c)$ for $c\sim 1$ from the Maxwellian ⇒ Transport coefficients

A brief (and incomplete) review of previous results

- Goldshtein & Shapiro (1995): First estimate of a_2 (linear aproximation neglecting a_2^2, a_3, a_4, ...) for d=3. Algebraic mistake.
- van Noije & Ernst (1998): Mistake corrected. Expression for general d.
- Brey, Ruiz-Montero & Cubero (1996): DSMC validation of vNE98's result for a_2 (d=3). DSMC computation of a_3 (d=3).
- Garzó & Dufty (1999): Linear approximation for a_2 (d=3) in a binary mixture.
- Montanero & Santos (2000): Ambiguity of the linear approximation for a_2 and expression alternative to vNE98's. DSMC computation of a_2 and a_3 (d=3).
- Brilliantov & Pöschel (2000): Cubic equation for a_2 (neglecting a_3, a_4, ...) in the case d=3.

A brief (and incomplete) review of previous results (cont.)

- Huthmann, Orza & Brito (2000): Assume that $a_k = O(\lambda^k)$. MD computation of a_2 (d=2).
- Montanero & Garzó (2002): DSMC validation of GD99's result for a_2 (d=3) in a binary mixture.
- Coppex, Droz, Piasecki & Trizac (2003): Extensive analysis on the ambiguity of the linear approximation for a_2. Alternative approach to estimate a_2. DSMC computation of a_2 (d=2).
- Brilliantov & Pöschel (2006): Linear approximation for a_2 and a_3 (neglecting a_2^2, a_2a_3, a_3^2, a_4, ...). DSMC computation of a_2-a_6 (d=3). Evidence of the divergent character of the Sonine expansion.
- Noskowicz, Bar-Lev, Serero & Goldhirsch (2007): Computer-aided method to evaluate (numerically) a_k. Fitted expression for a_2 (d=3). Confirmation of the divergence of the Sonine expansion.

Journal of Fluid Mechanics Digital Archive , Volume 282, January 1995, pp 75-114

$$a_2^{\mathrm{GS}} = \frac{16(1-\alpha)(1-2\alpha^2)}{401 - 337\alpha + 190(1-\alpha)\alpha^2}$$

81 17 30

(d=3)

Granular Matter 1, 57–64

$$a_2 = \frac{16(1-\alpha)(1-2\alpha^2)}{9 + 24d + 8\alpha d - 41\alpha + 30(1-\alpha)\alpha^2}$$

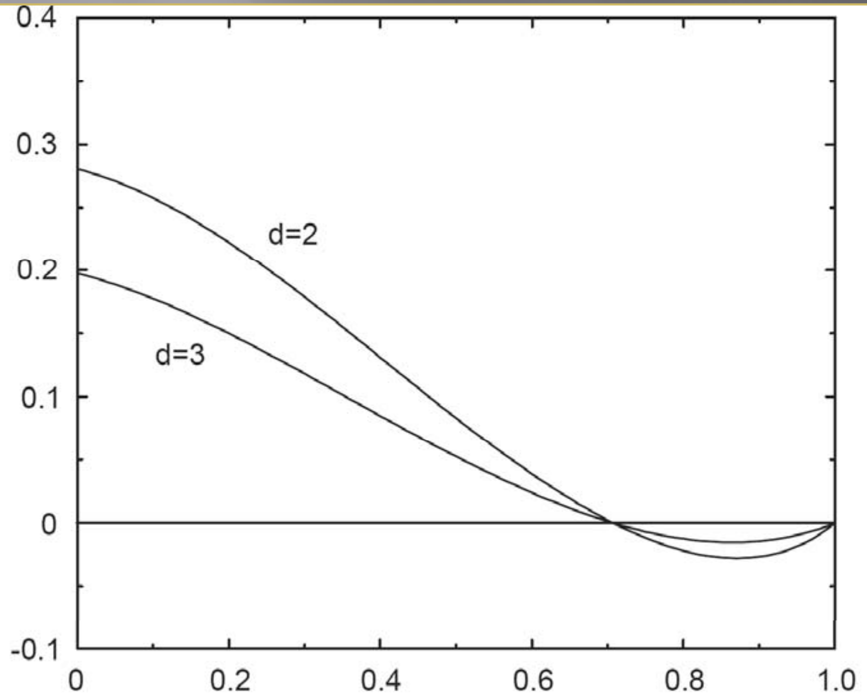


Fig. 1. Fourth cumulant a_2 versus α for homogeneous cooling solution in a freely evolving fluid

PHYSICAL REVIEW E VOLUME 54, NUMBER 4 3664 OCTOBER 1996

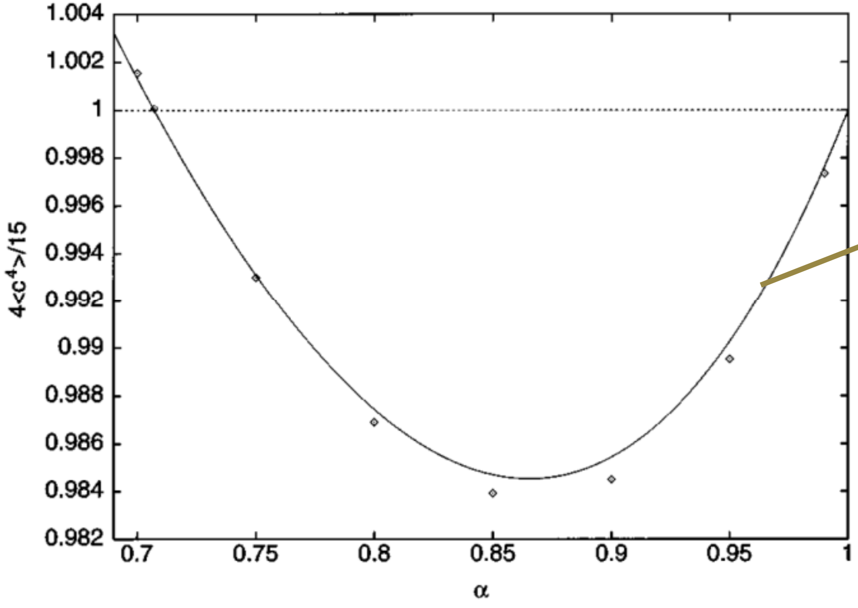


FIG. 5. Comparison of the value of the fourth moment in the HCS as obtained from the simulation (diamonds) and the theoretical prediction obtained in the first Enskog approximation (solid line).

vNE98

$(d=3)$

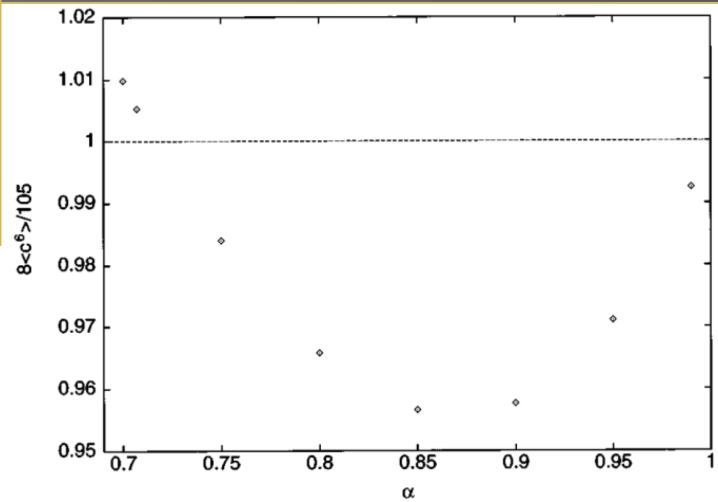


FIG. 6. Values of the sixth velocity moments in the HCS as a function of the restitution coefficient.

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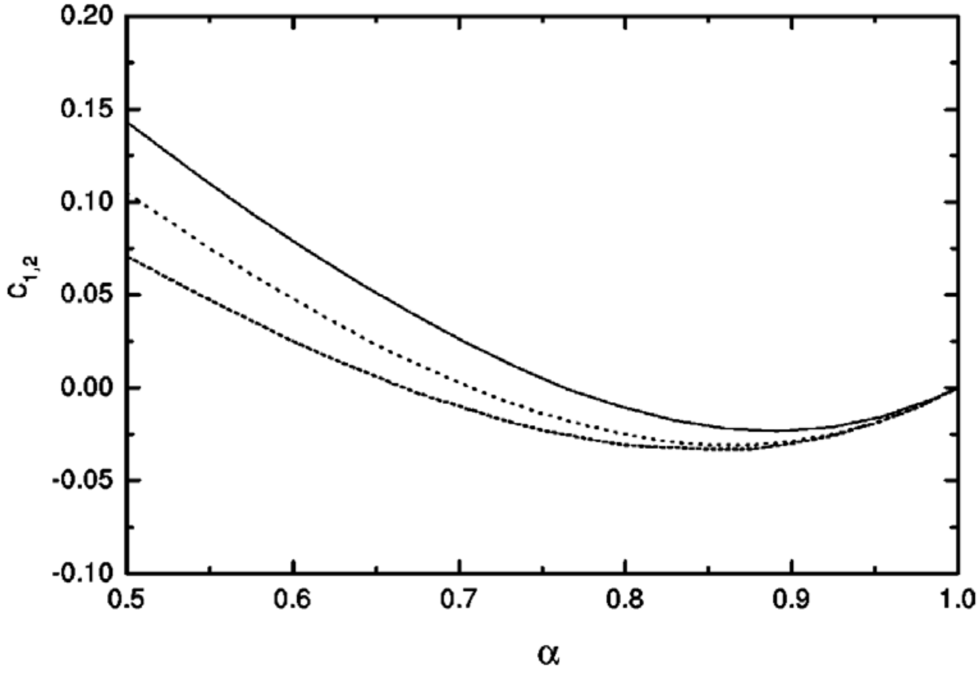

FIG. 1. Plot of the coefficients c_i versus the restitution coefficient $\alpha \equiv \alpha_{11} = \alpha_{22} = \alpha_{12}$ for $n^* = 0$, $\sigma_{11} = \sigma_{22} = \sigma_{12}$, $x_1/x_2 = 1$, and $m_1/m_2 = 2$. The solid line refers to c_1 while the dashed line corresponds to c_2. The dotted line is the common value in the single component case.

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$$a_2 \simeq -\frac{\mu_4^{(0)} - (d+2)\mu_2^{(0)}}{\mu_4^{(1)} - \mu_4^{(0)} - (d+2)\mu_2^{(1)}} = \frac{16(1-\alpha)(1-2\alpha^2)}{25 + 24d - \alpha(57 - 8d) - 2(1-\alpha)\alpha^2}$$

9 41 -30

$$a_2^{\mathrm{MS00}} = \frac{a_2^{\mathrm{vNE98}}}{1 + a_2^{\mathrm{vNE98}}}$$

vNE98

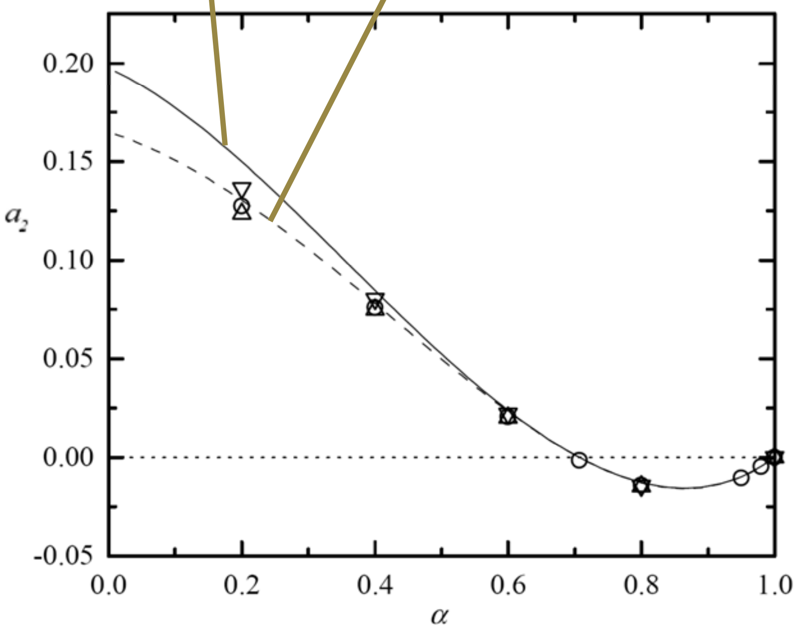

Fig. 4. Plot of the simulation values of a_2 (○), $(\mu_2 - \mu_2^{(0)})/\mu_2^{(1)}$ (△) and $(\mu_4 - \mu_4^{(0)})/\mu_4^{(1)}$ (▽) versus α in the case of the Gaussian thermostat. The solid and dashed lines are the theoretical estimates (5) and (40), respectively

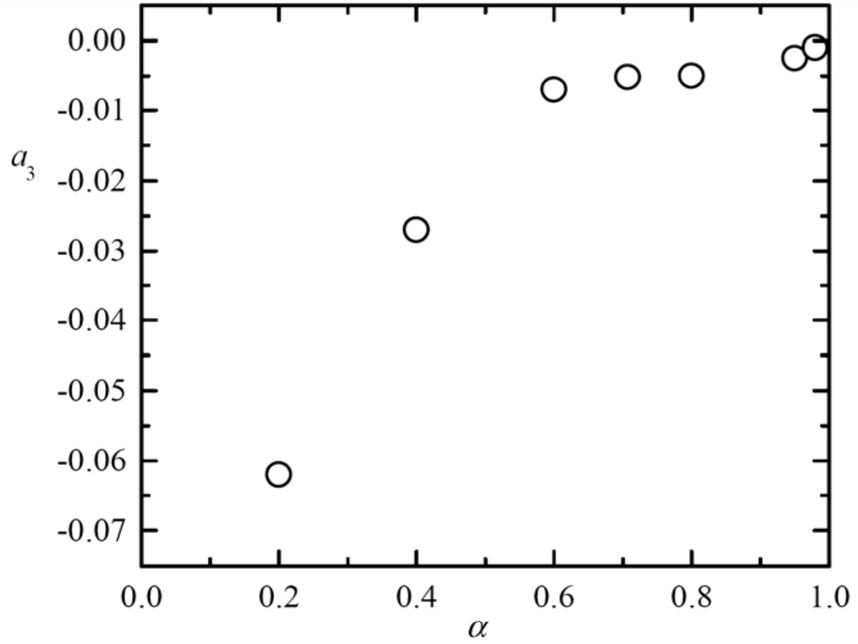

Fig. 6. Plot of the simulation values of a_3 versus α in the case of the Gaussian thermostat

(d=3)

PHYSICAL REVIEW E VOLUME 61, NUMBER 3 2809 MARCH 2000

$$a_2 = a_2^{\mathrm{NE}}\left[1 - \frac{1005(1-\epsilon^2) - 4096 T_3}{6080(1-\epsilon^2) - 4096 T_2} a_2^{\mathrm{NE}} + \cdots\right]$$

(d=3)

vNE98

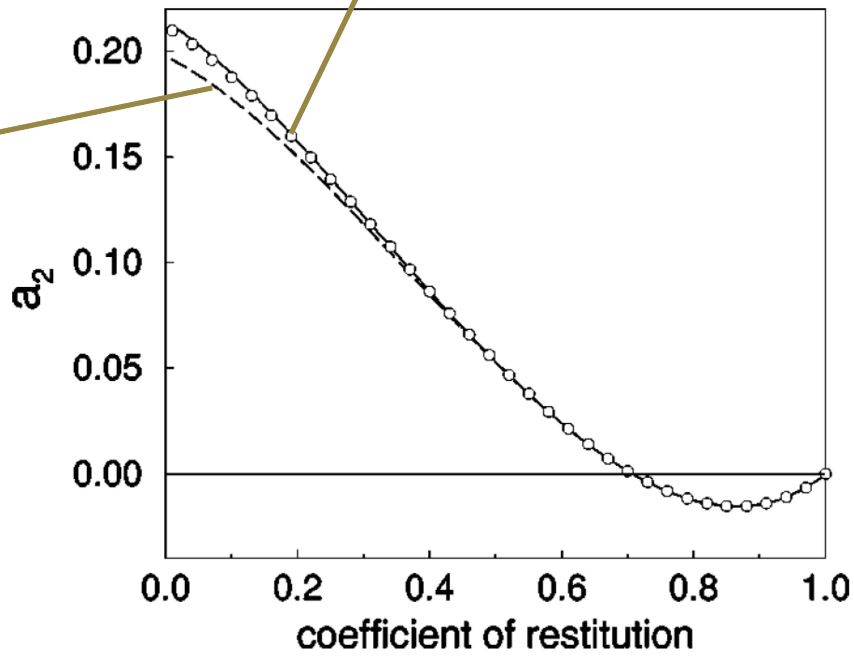

FIG. 2. The second Sonine coefficient a_2 as a function of the coefficient of restitution ϵ (full line). The dashed line shows a_2^{NE} in the first order approximation by van Noije and Ernst [3] according to Eq. (16). The approximation (17) is shown by circles.

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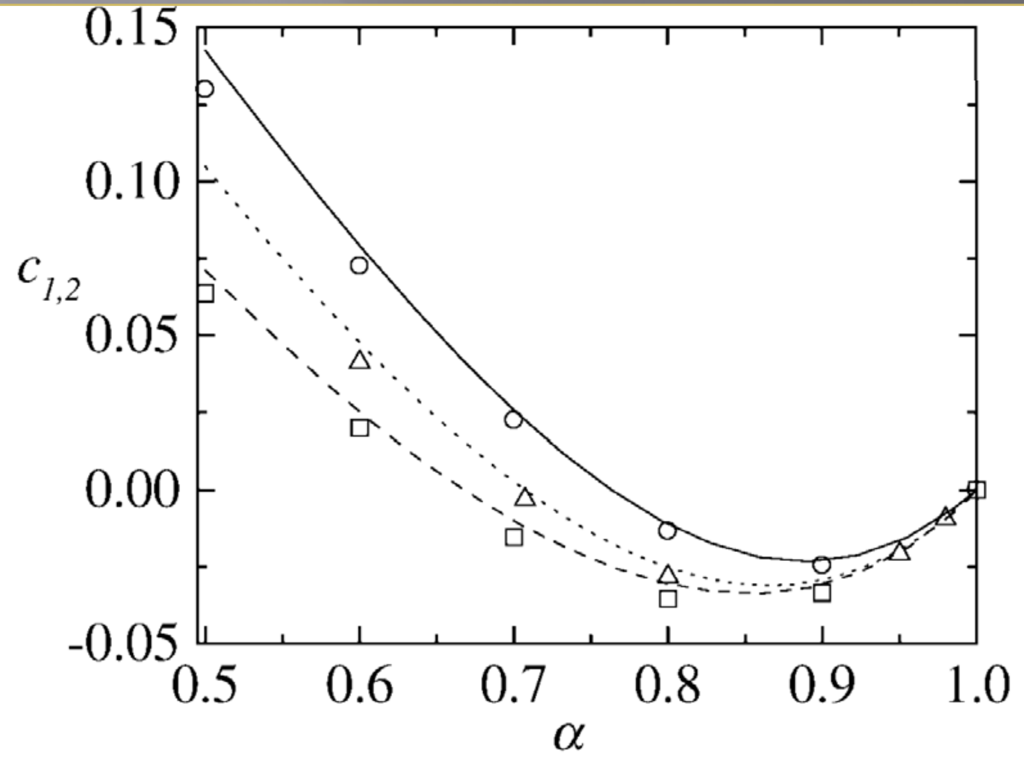

Fig. 2. Plot of the coefficients c_i versus the restitution coefficient α for $n^* = 0$, $\delta = 1$, $w = 1$ and $\mu = 2$. The solid line and the circles refer to c_1 while the dashed line and the squares correspond to c_2. The dotted line and the triangles refer to the common value in the single component case. The lines are the theoretical predictions and the symbols correspond to the simulation results

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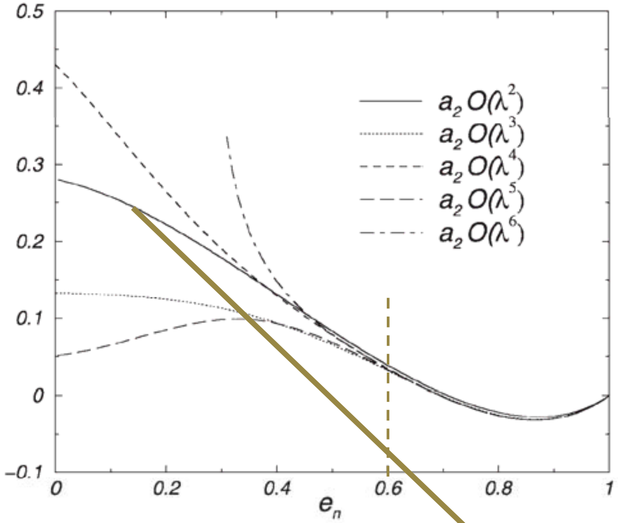


Fig. 7. Stationary value of a_2 to order $\mathcal{O}(\lambda^3)$ up to $\mathcal{O}(\lambda^6)$ as function of e_n

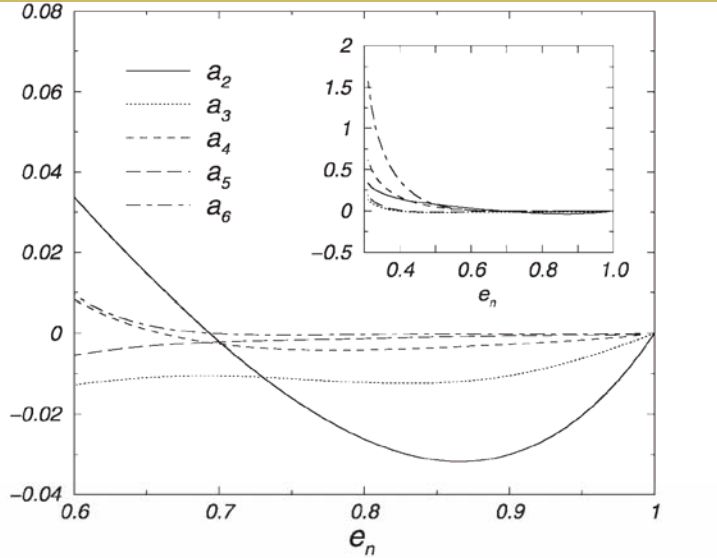


Fig. 6. Stationary values $a_2, \ldots, a_6$ calculated to order $\mathcal{O}(\lambda^6)$ as a function of e_n

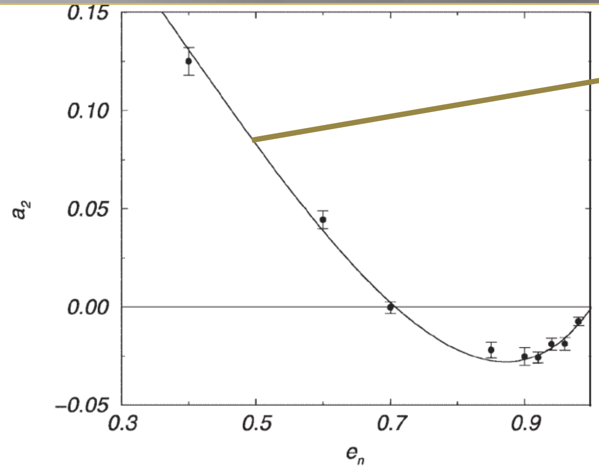


Fig. 10. Coefficient a_2 versus the coefficient of restitution e_n. The solid line is the theoretical prediction of Eq. (26) and the circles are the values calculated from MD simulations with their corresponding error bars

vNE98

(d=2)

Physica A 329 (2003) 114–126

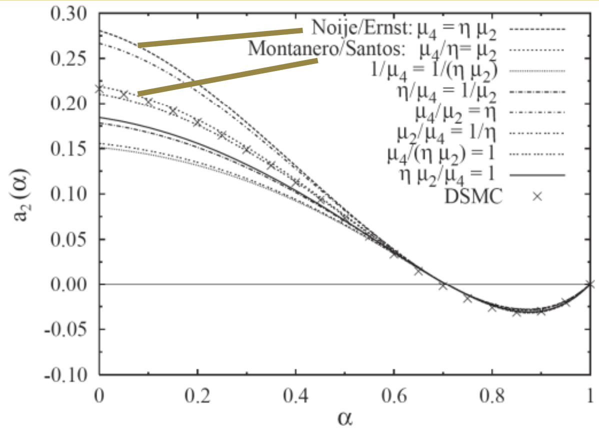


Fig. 5. The eight possible fourth cumulant a_2 obtained from Eq. (11), corresponding to the two-dimensional homogeneous free cooling. We define $\eta = (d+2)(1+a_2)$, then rewrite the equation $\mu_4 = \eta\mu_2$ according to the eight possible different combinations mentioned in the legend, before doing the linear Taylor expansion around $a_2 = 0$. The first curve is the plot of the function a_2 obtained by van Noije and Ernst [4], whereas the second one—obtained by Montanero and Santos [8] —is very close to the exact results shown by crosses.

(d=2)

vNE98

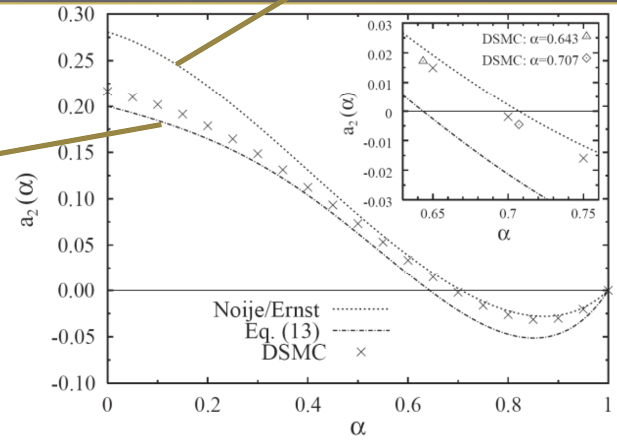


Fig. 1. Comparison of the correction $a_2(\alpha)$ for the free cooling in two dimensions obtained in Ref. [4], with Eq. (13). The crosses correspond to the "exact" result, obtained by solving the Boltzmann equation with the DSMC method, for 10^6 particles and approximately 500 collisions for each particle. The inset is a zoom in the region of the smallest root of the fourth cumulant.

$$a_2 = \frac{4(\alpha^2+1)^2(\alpha^2-1)[\sqrt{2}(\alpha^2+1)-2]}{A(\alpha, d)} ,$$

where

$$A(\alpha, d) = 5 + d(2-d) + 8\alpha(\alpha^2+1)(d-1) - \alpha^2(23 - 6d + d^2) + \alpha^4(3 + 6d + d^2) + \alpha^6(-1 + 2d + d^2) - \sqrt{2}(\alpha^2+1)^3(\alpha^2-1)(3 + 4d + 2d^2)/4 .$$

$$a_2 = -\frac{16}{b(\varepsilon)}\left(240\varepsilon^8 - 480\varepsilon^7 + 3312\varepsilon^6 - 7424\varepsilon^5 + 3510\varepsilon^4 - 364\varepsilon^3 + 895\varepsilon^2 + 1934\varepsilon - 1623\right),$$
$$a_3 = -\frac{128}{b(\varepsilon)}\left(80\varepsilon^8 - 160\varepsilon^7 + 816\varepsilon^6 - 1600\varepsilon^5 + 154\varepsilon^4 + 1548\varepsilon^3 - 669\varepsilon^2 - 386\varepsilon + 217\right), \qquad (16)$$
$$b(\varepsilon) = 2800\varepsilon^8 - 5600\varepsilon^7 + 34800\varepsilon^6 - 84480\varepsilon^5 - 4410\varepsilon^4 + 25716\varepsilon^3 + 112155\varepsilon^2 - 172458\varepsilon + 214357.$$

BP00 vNE98

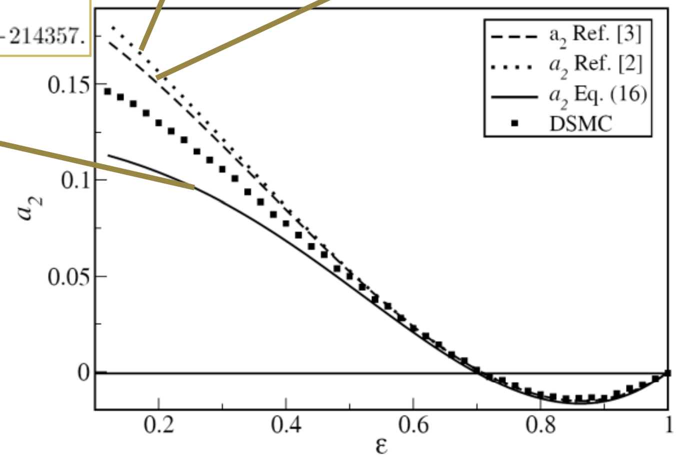


Fig. 1 – The second Sonine coefficient a_2 *vs.* the coefficient of restitution ε as given by eq. (16), where a_2 and a_3 are taken into account, $a_2(\varepsilon)$ from the linear theory, where only a_2 is taken into account [2], and $a_2(\varepsilon)$ from the corresponding non-linear theory [3], together with DSMC results.

(d=3)

Europhys. Lett., **74** (3), pp. 424–430 (2006)

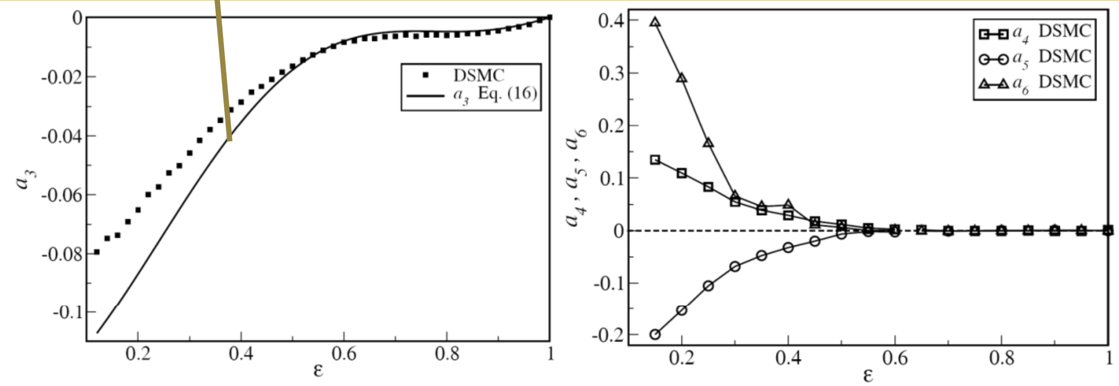


Fig. 2 – Left: the coefficient a_3 over the coefficient of restitution ε due to eq. (16) and to DSMC. Right: high-order Sonine coefficients as functions of ε (symbols). The lines guide the eye.

EPL, **79** (2007) 60001

$$a_2 = \frac{16(1-\alpha)(1-2\alpha^2)}{104.1 - 51.43\alpha + 78.67\alpha^2(1-\alpha)}.$$

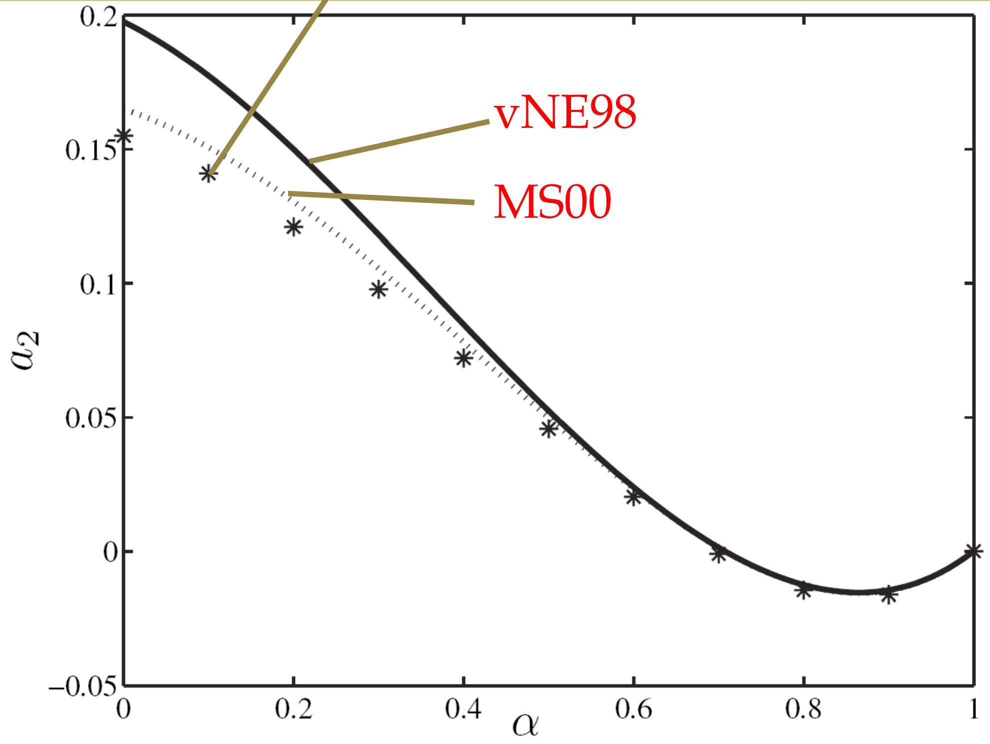


Fig. 3: A plot of the low-order result (solid line) and the result of [16] (hatched line) compared to the converged value of a_2 (asterisks) *vs.* α.

(d=3)

Can we derive theoretical expressions for a_2 and a_3 with an optimal compromise between simplicity and accuracy?

Try linear approximations!

(d=2)

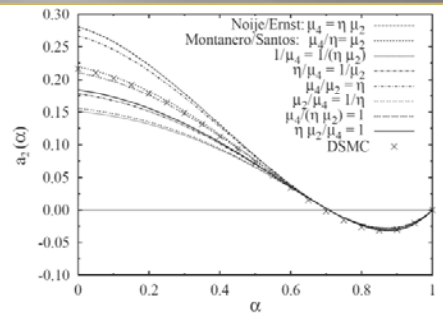


Fig. 5. The eight possible fourth cumulant a_2 obtained from Eq. (11), corresponding to the two-dimensional homogeneous free cooling. We define $\eta=(d+2)(1+a_2)$, then rewrite the equation $\mu_4=\eta\mu_2$ according to the eight possible different combinations mentioned in the legend, before doing the linear Taylor expansion around $a_2=0$. The first curve is the plot of the function a_2 obtained by van Noije and Ernst [4], whereas the second one—obtained by Montanero and Santos [8] —is very close to the exact results shown by crosses.

(d=3)

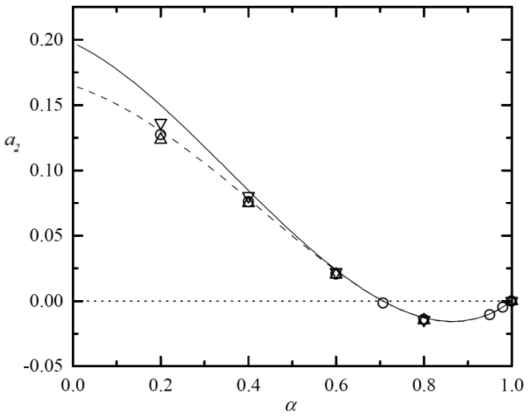


Fig. 4. Plot of the simulation values of a_2 (○), $(\mu_2-\mu_2^{(0)})/\mu_2^{(1)}$ (△) and $(\mu_4-\mu_4^{(0)})/\mu_4^{(1)}$ (▽) versus α in the case of the Gaussian thermostat. The solid and dashed lines are the theoretical estimates (5) and (40), respectively

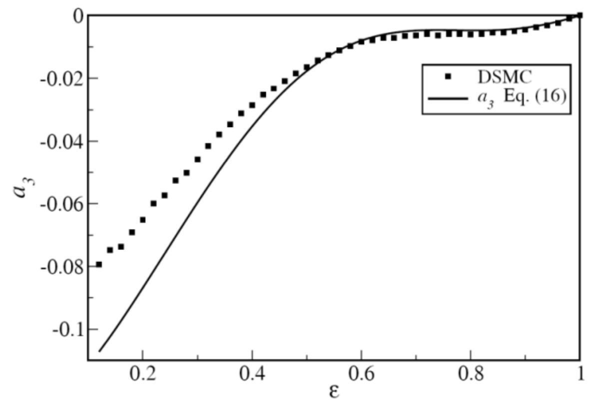


Fig. 2 – Left: the coefficient a_3 over the coefficie
Right: high-order Sonine coefficients as functions c

$$f(\mathbf{v},t) = n v_0^{-d}(t) F(c), \quad \mathbf{c}(t) = \frac{\mathbf{v}}{v_0(t)}$$

$$\partial_t f(\mathbf{v},t) = J[\mathbf{v}|f(t),f(t)] \Rightarrow \boxed{-\frac{\mu_2}{d}\frac{\partial}{\partial \mathbf{c}} \cdot \mathbf{c}F(\mathbf{c}) = I[\mathbf{c}|F,F]}$$

$$I[\mathbf{c}_1|F,F] = \int \mathrm{d}\mathbf{c}_2 \int \mathrm{d}\widehat{\boldsymbol{\sigma}}\, \Theta(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})(\mathbf{c}_{12}\cdot\widehat{\boldsymbol{\sigma}})\left[\alpha^{-2}F(\mathbf{c}_1^{**})F(\mathbf{c}_2^{**}) - F(\mathbf{c}_1)F(\mathbf{c}_2)\right]$$

Collisional moments: $\mu_p \equiv -\int \mathrm{d}\mathbf{c}\, c^p I[\mathbf{c}|F,F]$

Moment hierarchy: $\begin{cases} \text{(a)}\,\mu_p = \frac{p}{2}\mu_2\frac{\langle c^p\rangle}{\langle c^2\rangle}, & p \geq 4, \\ \text{(b)}\,\frac{\mu_p}{\langle c^p\rangle} = \frac{p}{2}\frac{\mu_2}{\langle c^2\rangle}, & p \geq 4. \end{cases}$

Linearizations

$$\mathcal{L}_{a_2}\{\mu_2\} = A_0(\alpha) + A_2(\alpha)a_2, \quad \mathcal{L}_{a_2,a_3}\{\mu_2\} = A_0(\alpha) + A_2(\alpha)a_2 + A_3(\alpha)a_3$$
$$\mathcal{L}_{a_2}\{\mu_4\} = B_0(\alpha) + B_2(\alpha)a_2, \quad \mathcal{L}_{a_2,a_3}\{\mu_4\} = B_0(\alpha) + B_2(\alpha)a_2 + B_3(\alpha)a_3$$
$$\mathcal{L}_{a_2}\{\mu_6\} = C_0(\alpha) + C_2(\alpha)a_2, \quad \mathcal{L}_{a_2,a_3}\{\mu_2\} = C_0(\alpha) + C_2(\alpha)a_2 + C_3(\alpha)a_3$$

Approximations

Class-I: Neglect a_3 in moment eq. for μ_4 (but not in moment eq. for μ_6)
Class-II: Treat a_2 and a_3 on the same footing (BP06)

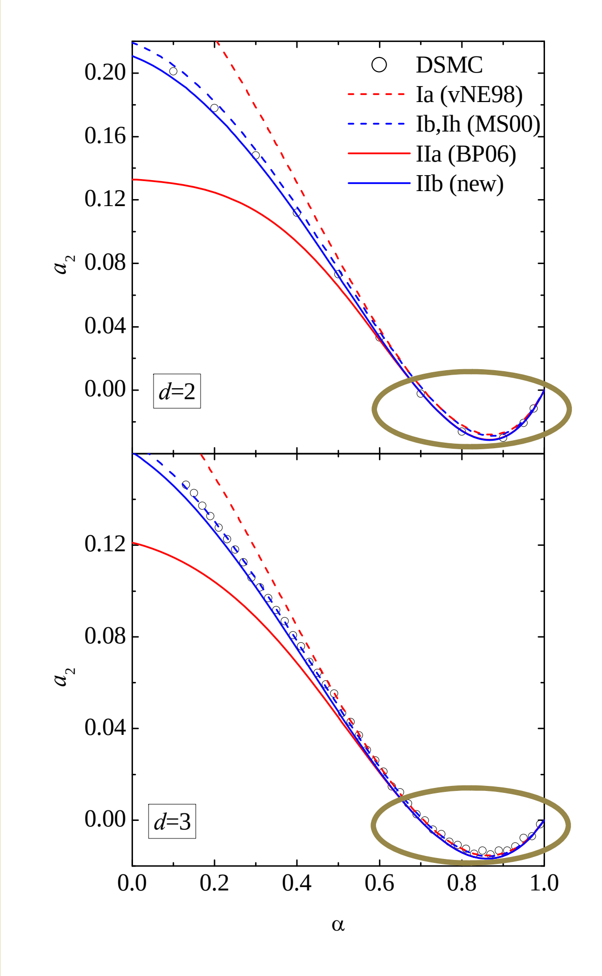
Approach (a) $\mathcal{L}\{\mu_p - \frac{p}{2}\mu_2\frac{\langle c^p\rangle}{\langle c^2\rangle}\} = 0$

Approach (b) $\mathcal{L}\{\frac{\mu_p}{\langle c^p\rangle} - \frac{p}{2}\frac{\mu_2}{\langle c^2\rangle}\} = 0$

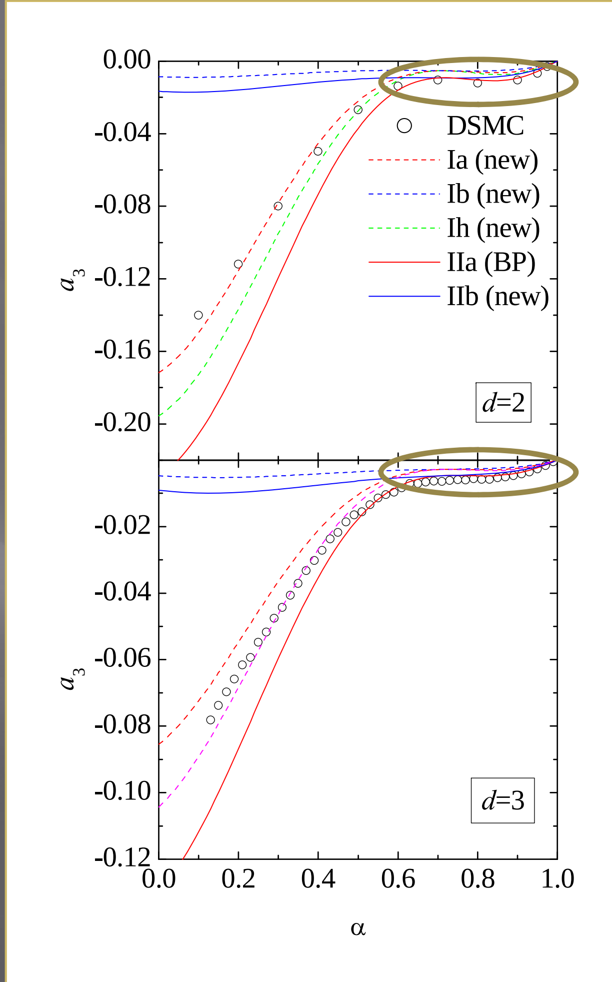
Linear approximations

Label	Equations	a_2	a_3
Ia	$L_2^{\mathrm{Ia}} \equiv \mathcal{L}_{a_2}\left\{\mu_4 - 2\mu_2\langle c^4\rangle/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6 - 3\mu_2\langle c^6\rangle/\langle c^2\rangle\right\} = 0$	vNE98	new
IIa	$L_2^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_4 - 2\mu_2\langle c^4\rangle/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6 - 3\mu_2\langle c^6\rangle/\langle c^2\rangle\right\} = 0$	BP06	BP06
Ib	$L_2^{\mathrm{Ib}} \equiv \mathcal{L}_{a_2}\left\{\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIb}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6/\langle c^6\rangle - 3\mu_2/\langle c^2\rangle\right\} = 0$	MS00	new
IIb	$L_2^{\mathrm{IIb}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIb}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6/\langle c^6\rangle - 3\mu_2/\langle c^2\rangle\right\} = 0$	new	new
Ih	$L_2^{\mathrm{Ib}} \equiv \mathcal{L}_{a_2}\left\{\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6 - 3\mu_2\langle c^6\rangle/\langle c^2\rangle\right\} = 0$	MS00	new

Comparison with DSMC simulations

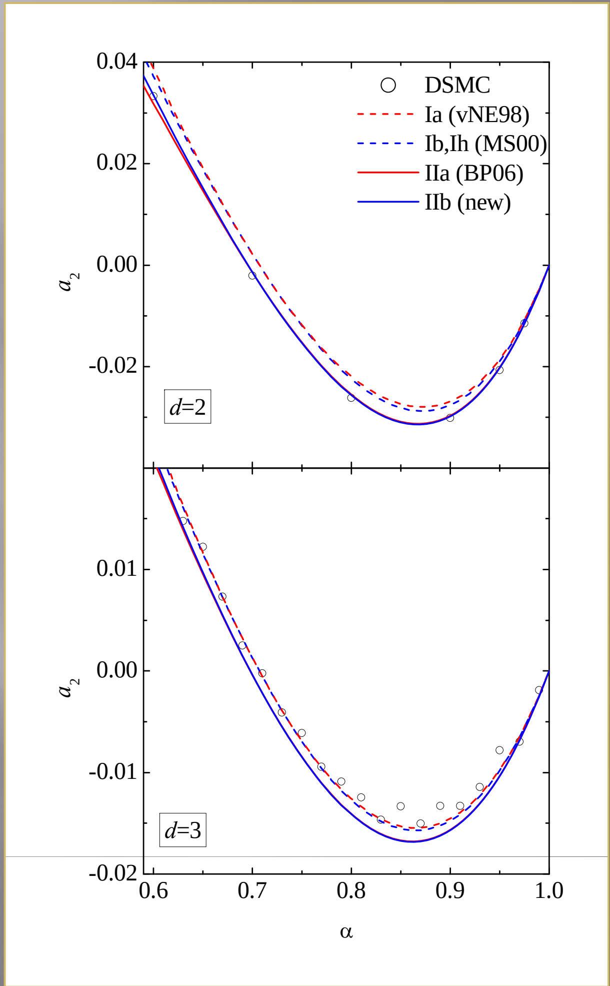


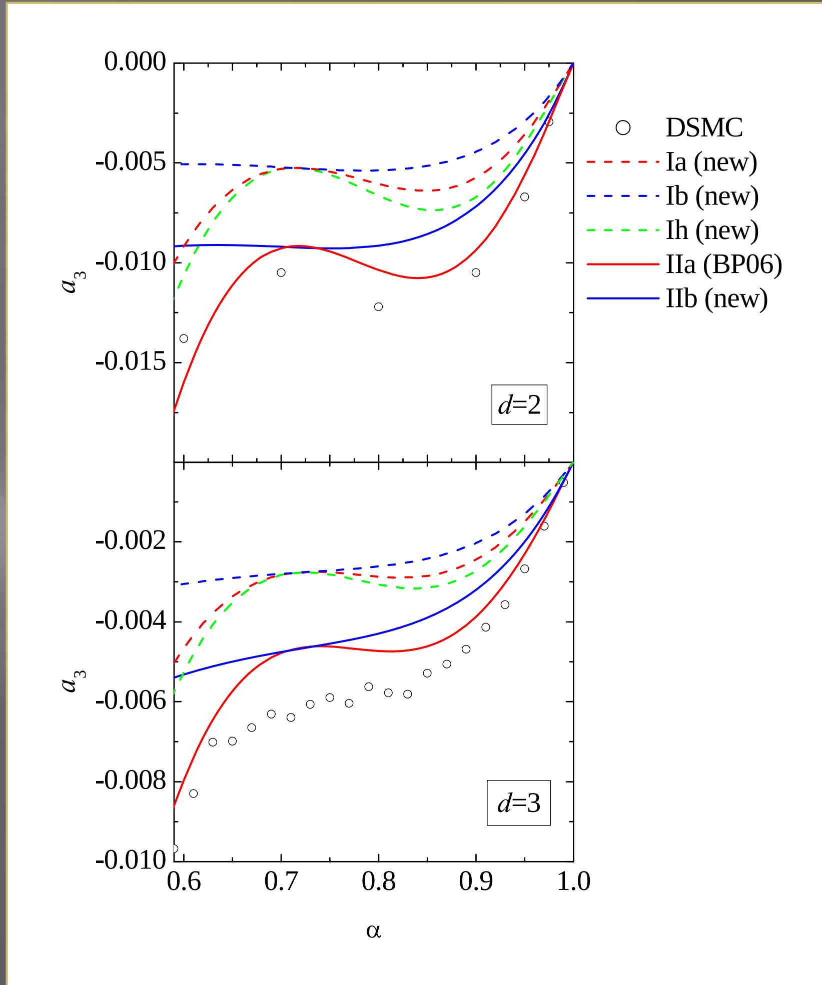
(New simulations)



(BP06 simulations)

Comparison with DSMC simulations





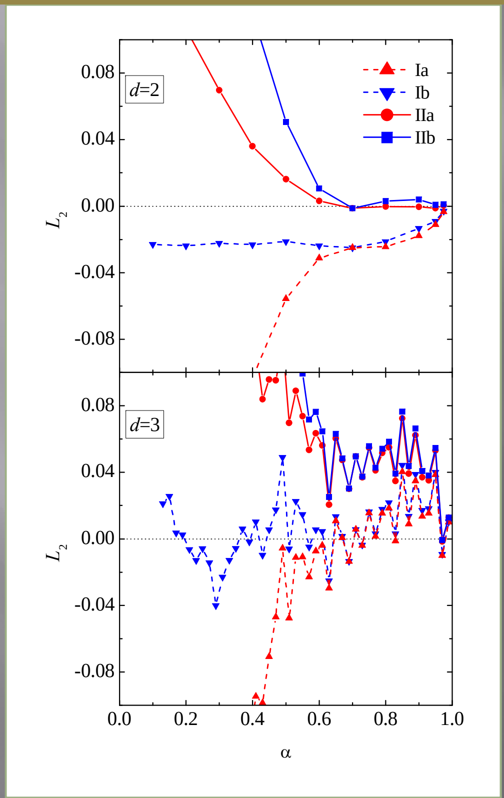
Assessment of the linear approximations

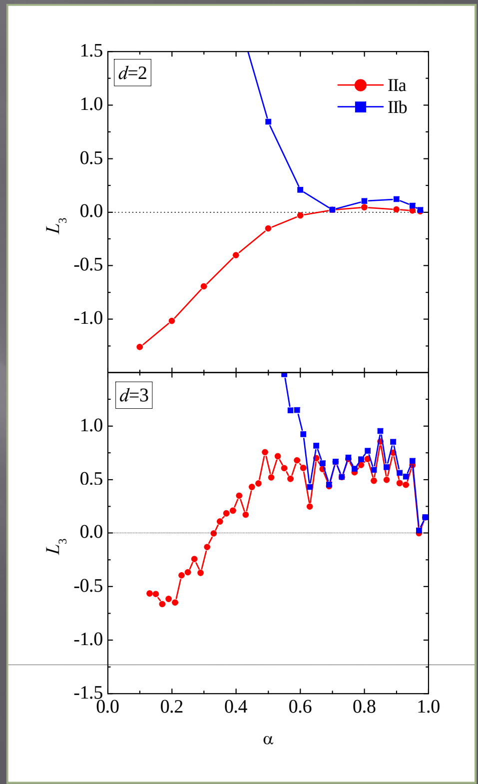
Label	Equations	a_2	a_3
Ia	$L_2^{\mathrm{Ia}} \equiv \mathcal{L}_{a_2}\left\{\mu_4 - 2\mu_2\langle c^4\rangle/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6 - 3\mu_2\langle c^6\rangle/\langle c^2\rangle\right\} = 0$	vNE98	new
IIa	$L_2^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_4 - 2\mu_2\langle c^4\rangle/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6 - 3\mu_2\langle c^6\rangle/\langle c^2\rangle\right\} = 0$	BP06	BP06
Ib	$L_2^{\mathrm{Ib}} \equiv \mathcal{L}_{a_2}\left\{\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIb}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6/\langle c^6\rangle - 3\mu_2/\langle c^2\rangle\right\} = 0$	MS00	new
IIb	$L_2^{\mathrm{IIb}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIb}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6/\langle c^6\rangle - 3\mu_2/\langle c^2\rangle\right\} = 0$	new	new
Ih	$L_2^{\mathrm{Ib}} \equiv \mathcal{L}_{a_2}\left\{\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle\right\} = 0$ $L_3^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6 - 3\mu_2\langle c^6\rangle/\langle c^2\rangle\right\} = 0$	MS00	new

$$L_2^{\mathrm{Ia}} \equiv \mathcal{L}_{a_2}\left\{\mu_4 - 2\mu_2\langle c^4\rangle/\langle c^2\rangle\right\} = 0, \quad \boxed{L_2^{\mathrm{Ib}} \equiv \mathcal{L}_{a_2}\left\{\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle\right\}},$$

$$L_2^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_4 - 2\mu_2\langle c^4\rangle/\langle c^2\rangle\right\}, \quad L_2^{\mathrm{IIb}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle\right\},$$

$$\boxed{L_3^{\mathrm{IIa}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6 - 3\mu_2\langle c^6\rangle/\langle c^2\rangle\right\}}, \quad L_3^{\mathrm{IIb}} \equiv \mathcal{L}_{a_2,a_3}\left\{\mu_6/\langle c^6\rangle - 3\mu_2/\langle c^2\rangle\right\}$$





Conclusions & Questions

- The (hybrid) linear approximation Ih (L_2^{Ib}=0, L_3^{IIa}=0) provides simple and accurate estimates for the general α-dependence of a_2 and a_3.
- However, if one needs a more precise estimate of a_3 in the region $0.6 \lesssim \alpha < 1$, the best choice is IIa≡BP06 (L_2^{IIa}=0, L_3^{IIa}=0).
- Even though a_2^2, a_3, a_4, ... are not negligible if $\alpha \lesssim 0.6$, they "conspire" to play a negligible role in $\mu_4/\langle c^4\rangle - 2\mu_2/\langle c^2\rangle$. Can we learn something of $F(c)$ by exploiting this?
- Frustration: Will we ever be able to get a closed form (even in terms of special functions, etc.) for $F(c)$?

Thank you for your attention!

α	0.8	0.6	0.4	0.2
a_2	-0.0141	0.0207	0.0760	0.1274
μ_2	0.8950	1.6101	2.1354	2.4625
$\mathcal{L}_{a_2}\{\mu_2\}$	0.9000	1.6105	2.1356	2.4639
$\mu_2(1+a_2)$	0.8824	1.6434	2.2976	2.7762
$\mathcal{L}_{a_2}\{\mu_2(1+a_2)\}$	0.8873	1.6436	2.2956	2.7705
μ_4	4.414	8.213	11.494	13.881
$\mathcal{L}_{a_2}\{\mu_4\}$	4.421	8.188	11.404	13.686
$\mu_4/(1+a_2)$	4.477	8.047	10.682	12.312
$\mathcal{L}_{a_2}\{\mu_4/(1+a_2)\}$	4.487	8.027	10.658	12.294