# DSMC Evaluation of the Navier-Stokes Shear Viscosity of a Granular Fluid

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# System:

- (Monodisperse) granular fluid of inelastic hard spheres with packing fraction $\phi$ and coefficient of normal restitution $\alpha$.
- Navier-Stokes (NS) constitutive equations:

$$\mathbf{q} = -\kappa \nabla T - \mu \nabla n$$

Thermal conductivity

“Diffusion thermo-effect”

$$P_{ij} = -\eta \left( \nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij} \nabla \cdot \mathbf{u} \right) - \eta_B \delta_{ij} \nabla \cdot \mathbf{u}$$

Shear viscosity

Bulk viscosity

- By dimensional analysis, the temperature dependence of the transport coefficients is

$$\kappa \propto \mu/T \propto \eta \propto \eta_B \propto T^{1/2}$$

- Otherwise, they are nonlinear functions of $\alpha$ and $\phi$:

$$\kappa(\alpha,\phi), \quad \mu(\alpha,\phi), \quad \eta(\alpha,\phi), \quad \eta_B(\alpha,\phi)$$

- Assuming the validity of the Enskog equation (*stosszahlansatz*), the NS transport coefficients can be derived from the Chapman-Enskog method around the *homogeneous cooling state* (HCS) (Garzó & Dufty, 1999).
- As in the elastic case, in order to get explicit expressions, one expands the NS distribution function $f^{(1)}(\mathbf{V})$ in Sonine polynomials and truncates the series.

- In particular, in the case of the shear viscosity,

$$f^{(1)}(\mathbf{V}) \to -\frac{m\eta^k}{nT^2} f_M(\mathbf{V}) \left(V_iV_j - \frac{2}{3}\delta_{ij}V^2\right) \nabla_i u_j$$

First Sonine approximation

$$\eta^k(\alpha,\phi) = \eta_0 \frac{1-\frac{2}{5}\phi\chi(\phi)(1+\alpha)(1-3\alpha)}{\frac{1}{384}\chi(\phi)(1+\alpha)\left[16(13-\alpha)-3(4-3\alpha)c_0(\alpha)\right]}$$

Kinetic part

$$\eta(\alpha,\phi) = \eta^k(\alpha,\phi)\left[1+\frac{4}{5}\phi\chi(\phi)(1+\alpha)\right] + \eta_0\frac{384}{25\pi}\phi^2\chi(\phi)(1+\alpha)\left[1-\frac{1}{32}c_0(\alpha)\right]$$

Total

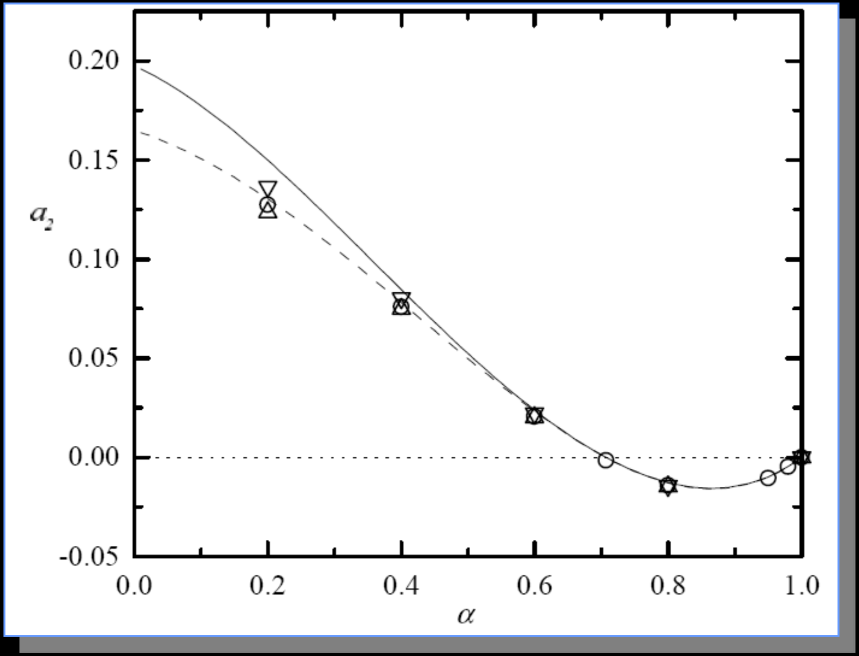
$$\underbrace{\phi = \frac{\pi}{6}n\sigma^3}_{\text{Packing fraction}}, \quad \underbrace{\chi(\phi) = \frac{1-\phi/2}{(1-\phi)^3}}_{\text{Carnahan-Starling}}, \quad \underbrace{\eta_0 = \frac{5}{16\sigma^2}\sqrt{\frac{mT}{\pi}}}_{\text{Dilute gas in the elastic limit}}, \quad \underbrace{c_0(\alpha) = \frac{32(1-\alpha)(1-2\alpha^2)}{81-17\alpha+30\alpha^2(1-\alpha)}}_{\text{Fourth velocity cumulant of the HCS}}$$

$(c=6\langle V^4\rangle/5\langle V^2\rangle^2-2)$

# How accurate is the first Sonine approximation for $\eta$?

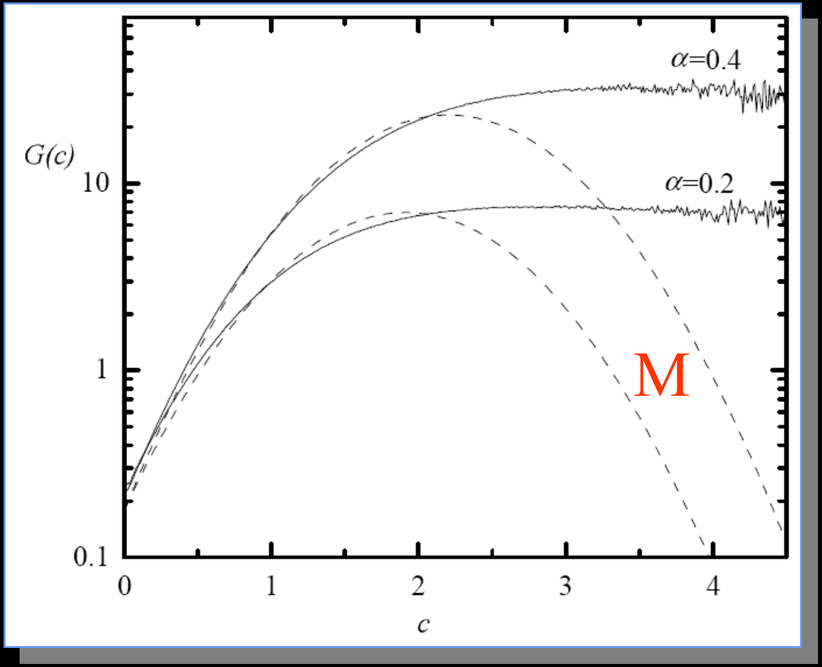
- In the elastic ($\alpha=1$) and dilute ($\phi\to 0$) limit it is known that the first Sonine approximation underestimates the shear viscosity by a 1.6%.
- It might be that the Sonine aproximation worsens with increasing inelasticity since the reference state (HCS) differs from a Maxwellian if $\alpha<1$:

## Non-zero fourth cumulant

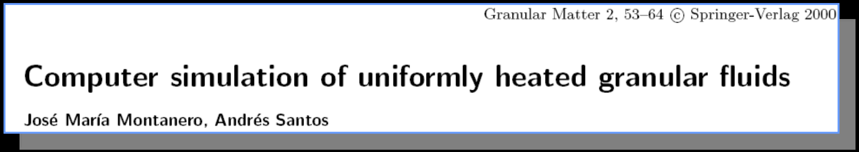


$$a_2=c_0/2$$

## High-velocity overpopulated tails



$$G(V) = f(V)e^{AV}$$


Granular Matter 2, 53–64 © Springer-Verlag 2000

**Computer simulation of uniformly heated granular fluids**

José María Montanero, Andrés Santos

To test the first Sonine approximation for η one needs to compute it *numerically* by DSMC simulations for a wide range of values of α and ϕ.

- An elegant method consists of analyzing the time evolution of a *weak* perturbation of the HCS under the form of a transverse shear wave (Brey *et al*., 1999):

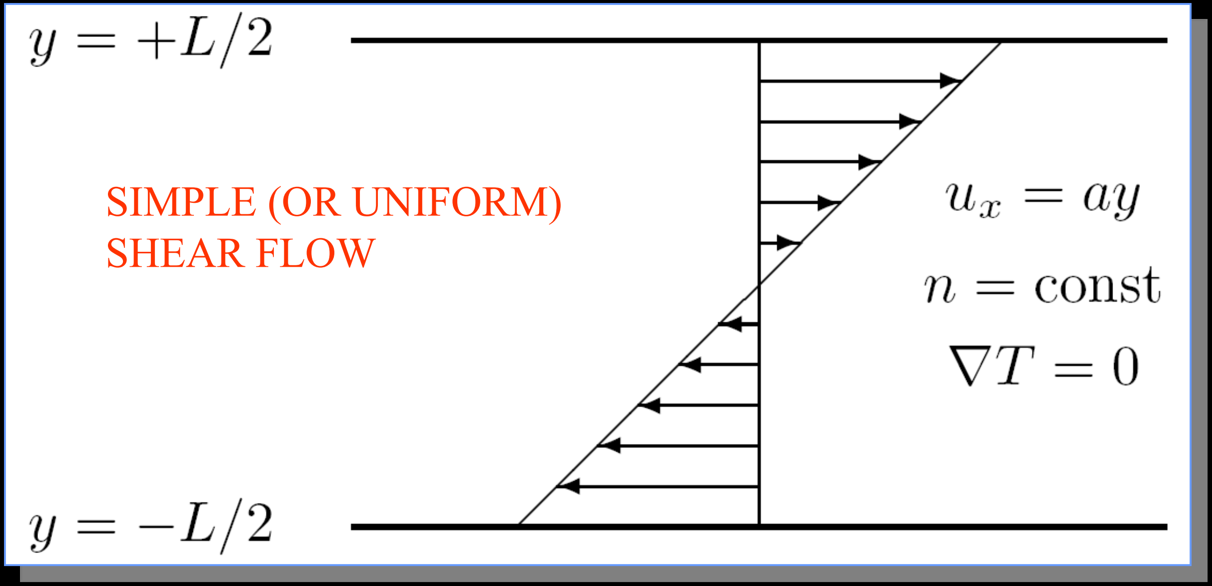
$$\frac{\partial u_x}{\partial t} = \frac{\eta(t)}{\rho}\frac{\partial^2 u_x}{\partial y^2} \Rightarrow u_x(y,t) = u_0 \sin(k_0 y) \exp\left[-\boxed{\frac{\eta(t)}{\rho\sqrt{T(t)}}} k_0^2 \int_0^t dt' \sqrt{T(t')}\right]$$

- Other method (so far, restricted to the dilute limit): Green-Kubo formulas (Dufty & Brey, 2002; Brey *et al.*, 2004).

See Brey’s talk for details and results.

# OUR GOAL:

- Propose a method to measure the Navier-Stokes shear viscosity $\eta(\alpha,\phi)$ from DSMC simulations of a *modified* simple shear flow.



# Enskog equation for the simple shear flow state:

shear rate

Lagrangian frame: $f\,(\mathbf{r},\mathbf{v},t)=f\,(\mathbf{V}(\mathbf{r}),t)$, $\mathbf{V}(\mathbf{r})=\mathbf{v}-\mathbf{u}(\mathbf{r})$, $\mathbf{u}(\mathbf{r})=\mathbf{a}.\mathbf{r}$, $a_{ij}=a\delta_{ix}\delta_{jy}$

$$\partial_t f - aV_y \frac{\partial}{\partial V_x} f + F[f] = J[f,f]$$

$$J[f,f]=\sigma^2\chi\int d\mathbf{V}_1\int d\widehat{\boldsymbol{\sigma}}\,\Theta(\widehat{\boldsymbol{\sigma}}\cdot\mathrm{g})(\widehat{\boldsymbol{\sigma}}\cdot\mathrm{g})[\alpha^{-2}f(\mathbf{V}')f(\mathbf{V}_1'+\underline{\mathsf{a}\cdot\boldsymbol{\sigma}})-f(\mathbf{V})f(\mathbf{V}_1-\underline{\mathsf{a}\cdot\boldsymbol{\sigma}})]$$

$$\mathbf{g}=\mathbf{V}-\mathbf{V}_1,\quad \mathbf{V}'=\mathbf{V}-\frac{1}{2}(1+\alpha^{-1})(\widehat{\boldsymbol{\sigma}}\cdot\mathrm{g})\widehat{\boldsymbol{\sigma}},\quad \mathbf{V}_1'=\mathbf{V}_1+\frac{1}{2}(1+\alpha^{-1})(\widehat{\boldsymbol{\sigma}}\cdot\mathrm{g})\widehat{\boldsymbol{\sigma}}$$

Inertial force

$F\,[f]$: Operator representing an “external” action (to be determined)

# Knudsen number of the problem

$$\underbrace{\mathrm{Kn}(t) = \frac{\lambda}{\ell(t)},}_{\text{Knudsen number}} \quad \underbrace{\lambda = \frac{6\sigma}{\sqrt{2}\phi\chi(\phi)},}_{\text{Mean free path}} \quad \underbrace{\ell(t) = \frac{\sqrt{2T(t)/m}}{a}}_{\text{Hydrodynamic length}}$$

Navier-Stokes regime $\Longrightarrow \mathrm{Kn}(t) \to 0 \Longrightarrow T(t) \to \infty$

Temperature must monotonically increase in time!

# Energy balance equation

$$\partial_t T = \underbrace{\frac{2}{3n} a|P_{xy}|}_{\text{Viscous heating}} \underbrace{- \zeta T}_{\text{Inelastic cooling}} \underbrace{+ \gamma T}_{\text{External heating}}$$

$$\underbrace{\zeta = \frac{m\sigma^2}{12nT}(1-\alpha^2)\chi \int d\widehat{\boldsymbol{\sigma}} \int d\mathbf{V} \int d\mathbf{V}_1 \Theta(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})^3 f(\mathbf{V} + \underline{\mathsf{a} \cdot \boldsymbol{\sigma}}) f(\mathbf{V}_1),}_{\text{Cooling rate (internal)}} \quad \underbrace{\gamma = -\frac{m}{3nT} \int d\mathbf{V} F[f]}_{\text{Heating rate (external)}}$$

- To frustrate a steady state, $F[f]$ must be such that $\gamma=\zeta$.
- In addition we require that, to first order in Kn, the Enskog equation for the (modified) simple shear flow coincides with the Chapman-Enskog solution:

$$f(\mathbf{V})=f^{(0)}(\mathbf{V})+f^{(1)}(\mathbf{V})+O(\mathrm{Kn}^2)$$

$f^{(0)}(\mathbf{V})$=HCS$\Longrightarrow$

$$F[f^{(0)}]=\frac{1}{2}\zeta^{(0)}\frac{\partial}{\partial\mathbf{V}}\cdot\left[\mathbf{V}f^{(0)}(\mathbf{V})\right]$$

$f^{(1)}(\mathbf{V})$=NS$\Longrightarrow$

$$F[f^{(1)}]=-\zeta^{(0)}T\partial_T f^{(1)}=\frac{1}{2}\zeta^{(0)}\frac{\partial}{\partial\mathbf{V}}\cdot\left[\mathbf{V}f^{(1)}(\mathbf{V})\right]+\frac{1}{2}\zeta^{(0)}f^{(1)}(\mathbf{V})$$

Natural choice:

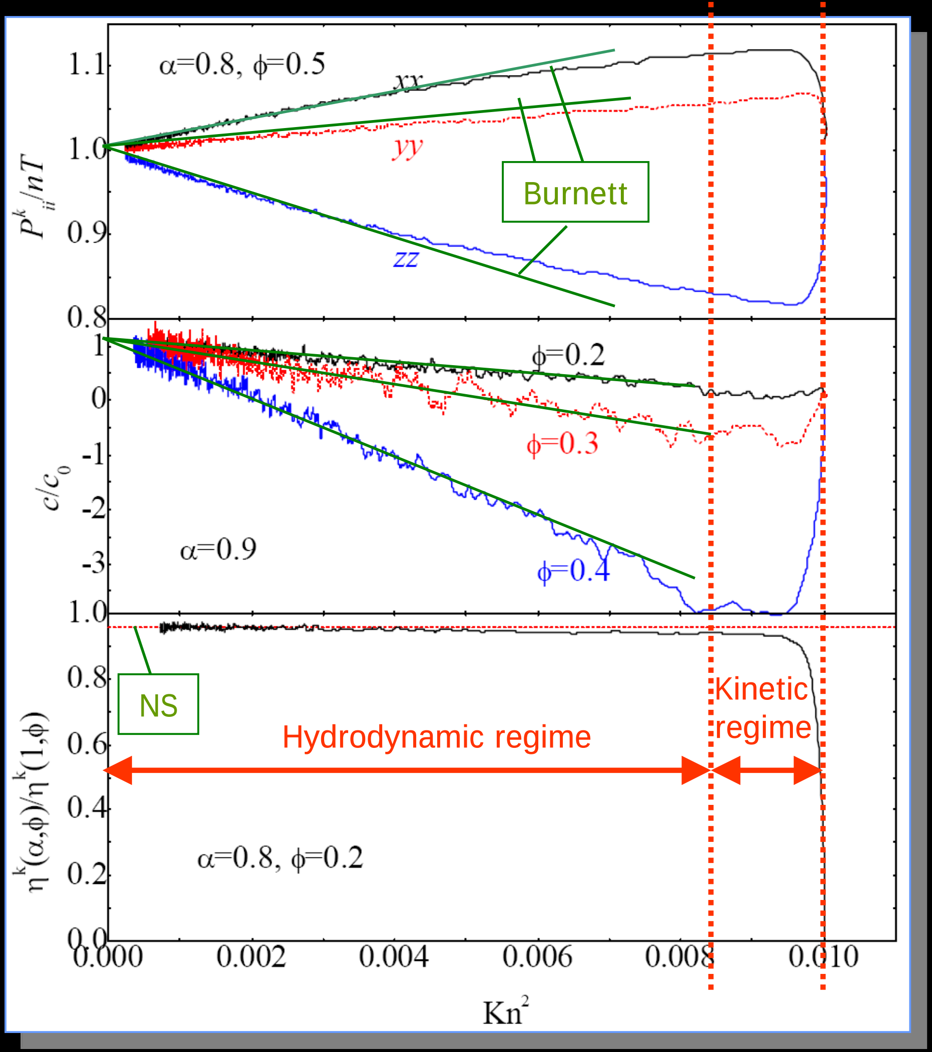
$$F[f]=\underbrace{\frac{1}{2}\zeta\frac{\partial}{\partial\mathbf{V}}\cdot(\mathbf{V}f)}_{\text{Gaussian “thermostat”}}+\underbrace{\frac{1}{2}\zeta\left(f-f^{(0)}\right)}_{\text{BGK-like relaxation term}}$$

# DSMC implementation

- Uniform system (Lagrqangian frame): No cells
- Collision stage: standard, but adapted to the inelastic Enskog collision operator.
- Free straming stage:

1. Gaussian thermostat: $\mathbf{V} \rightarrow \mathbf{V} \exp(\zeta\, \delta t/2)$.
2. Inertial force: $V_x \rightarrow V_x - a\, \delta t\, V_y$.
3. BGK-like relaxation: With probability $\zeta\, \delta t/2$, the velocity $\mathbf{V}_{\mathrm{old}}$ of every particle is replaced by a new velocity $\mathbf{V}_{\mathrm{new}}$ sampled from the HCS distribution $f^{(0)}(\mathbf{V})$.

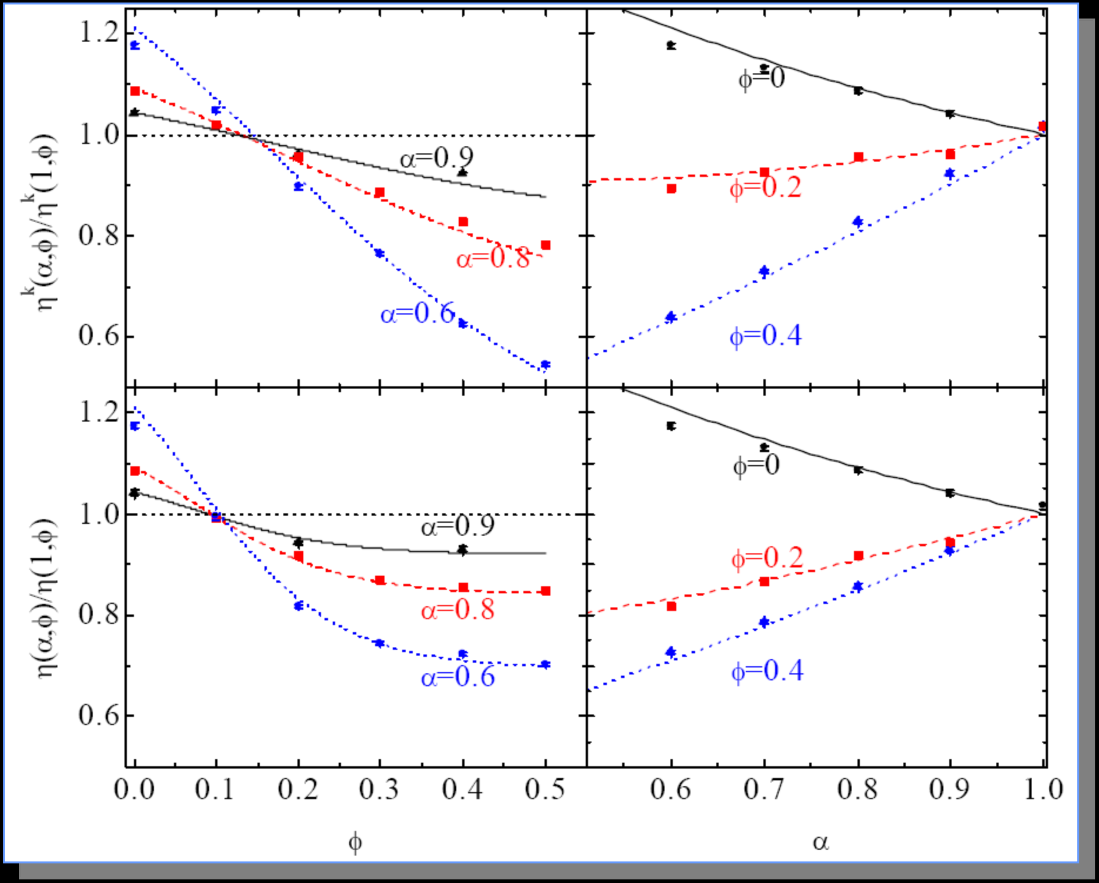
# Results (Evolution)



# Results (NS shear viscosity)

Kinetic viscosity

Total viscosity



$\eta(\alpha,\phi)>\eta(1,\phi)$ if $\phi \lesssim 0.1$; $\eta(\alpha,\phi)<\eta(1,\phi)$ if $\phi \gtrsim 0.1$

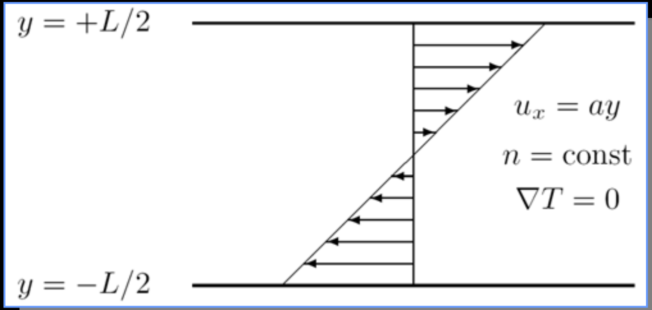
# Conclusions (I)

- An efficient simulation method to measure the NS shear viscosity of a dense granular fluid has been proposed.
- After a short transient period, the system reaches first a non-Newtonian hydrodynamic regime, then a Navier-Stokes regime, and finally tends asymptotically to the HCS (even with an increasing temperature!).

# Conclusions (II)

- The results show that the first Sonine approximation is as reliable as in the elastic limit, at least for this transport coefficient.
- A similar approach can be applied to the evaluation of the heat transport coefficients, where the Sonine approximation might be less accurate.

# THANKS!



$$\partial_t f - aV_y \frac{\partial}{\partial V_x} f + F[f] = J[f,f]$$

$$F[f] = \frac{1}{2}\zeta \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f) + \frac{1}{2}\zeta \left( f - f^{(0)} \right)$$