ON THE CONVERGENCE OF THE CHAPMAN-ENSKOG EXPANSION FOR GRANULAR GASES

Andrés Santos
Departamento de Física,
Universidad de Extremadura,
Badajoz (Spain)

STATISTICAL PHYSICS in EXTREMADURA

Outline

- Hydrodynamic description in ordinary gases.
- The Uniform Shear Flow and the Uniform Longitudinal Flow.
- Convergence/Divergence of the Chapman-Enskog expansion.
- Conclusions.

Hydrodynamic description in *ordinary* gases

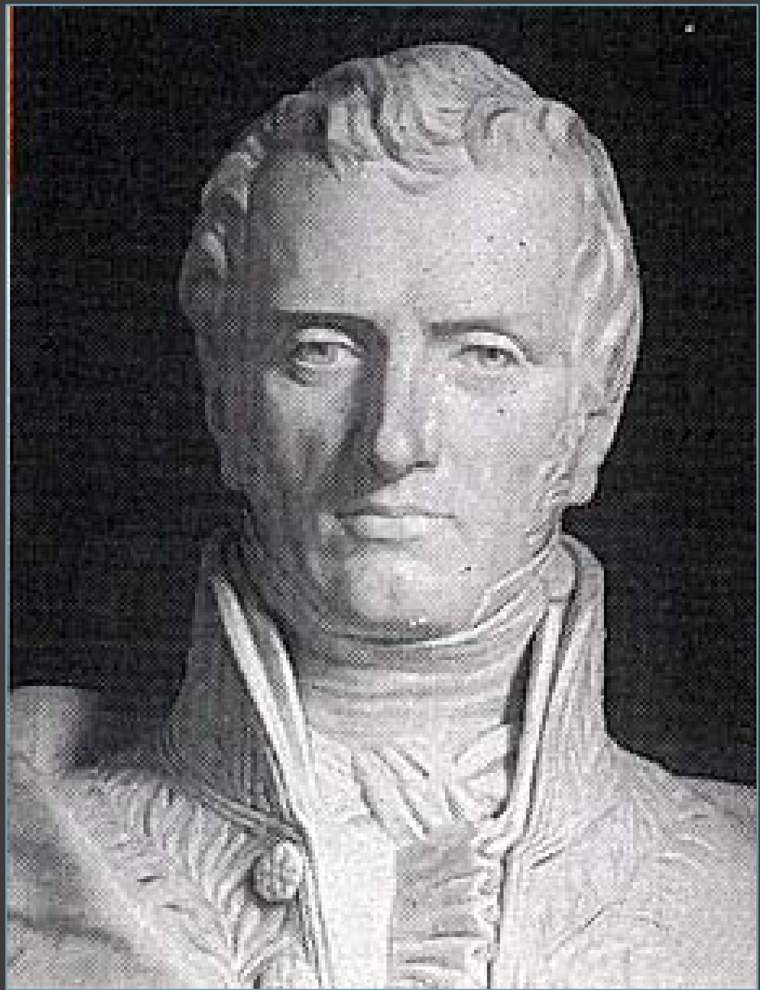
• Conservation equations (mass, momentum, and energy):

$$\underbrace{\partial_t y_i(\mathbf{r},t)}_{\text{Hydrodynamic fields}} + \nabla \cdot \underbrace{\mathbf{J}_i(\mathbf{r},t)}_{\text{Fluxes}} = 0$$

• Constitutive equations:

$$\mathbf{J}_i(\mathbf{r},t) = \mathcal{F}_i[\{y_j\}]$$

Closed set of equations



Claude-Louis Navier
(1785-1836)

George Gabriel Stokes
(1819-1903)

Navier-Stokes constitutive equations

$$P_{ij} = p\delta_{ij} - \eta_0\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\nabla \cdot \mathbf{u}\,\delta_{ij}\right)$$

Stress tensor

Viscosity

Newton's law

$$\mathbf{q} = -\kappa_0 \nabla T$$

Heat flux

Thermal conductivity

Fourier's law

Mean free path $\ell \ll L_h$ Hydrodynamic length

Hydrodynamics beyond Navier-Stokes: the Chapman-Enskog method

Sydney Chapman
(1888-1970)

David Enskog
(1884-1947)

Chapman-Enskog expansion:

$$\mu \sim \frac{\ell}{L_h} \sim \nabla: \text{ uniformity parameter}$$

$$\begin{array}{ccc|c|c|c} P_{ij} & = & p\delta_{ij} & +\mu P_{ij}^{(1)} & +\mu^2 P_{ij}^{(2)} & +\cdots \\ \mathbf{q} & = & \mathbf{0} & +\mu \mathbf{q}^{(1)} & +\mu^2 \mathbf{q}^{(2)} & +\cdots \end{array}$$

Euler	Navier-Stokes	Burnett

Non-Newtonian behavior

Incompressible flow $$P_{xy} = -\sum_{k=0}^{\infty} \eta_k \left(\frac{\partial u_x}{\partial y}\right)^{2k+1} + \cdots$$

Compressible flow $$P_{xx} = p - \frac{4}{3}\sum_{k=0}^{\infty} \eta'_k \left(\frac{\partial u_x}{\partial x}\right)^{k+1} + \cdots$$

$\eta_0 = \eta'_0$: NS, η'_1: Burnett, η_1, η'_2: super-Burnett, $\cdots$

Are the (*partial*) CE series

$$P_{xy} = -\sum_{k=0}^{\infty} \eta_k \left(\frac{\partial u_x}{\partial y} \right)^{2k+1}$$

$$P_{xx} = p - \frac{4}{3} \sum_{k=0}^{\infty} \eta'_k \left(\frac{\partial u_x}{\partial x} \right)^{k+1}$$

convergent?

Do there exist states where …?

$$P_{xy} = -\sum_{k=0}^{\infty} \eta_k \left(\frac{\partial u_x}{\partial y} \right)^{2k+1} + \cdots$$

$$P_{xx} = p - \frac{4}{3} \sum_{k=0}^{\infty} \eta'_k \left(\frac{\partial u_x}{\partial x} \right)^{k+1} + \cdots$$

YES! The Uniform Shear Flow and the Uniform Longitudinal Flow

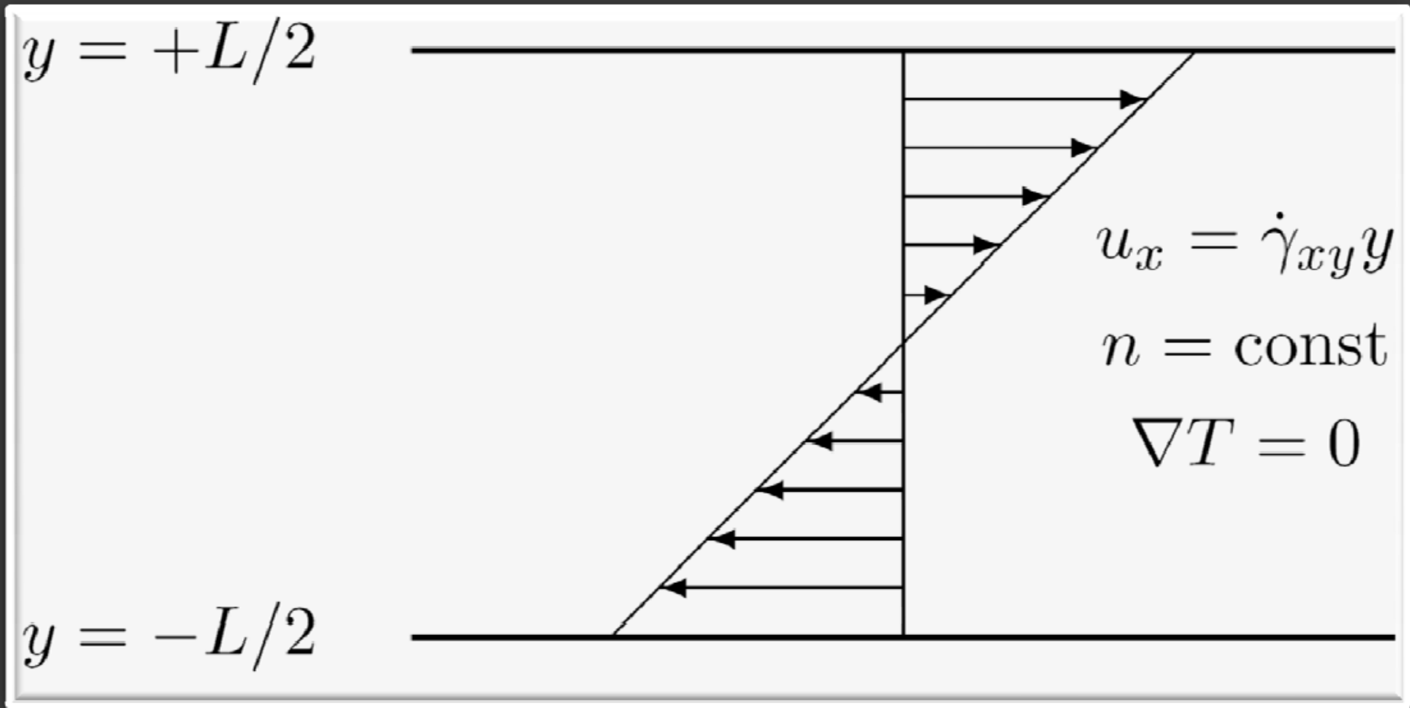
$$\nabla_i u_j = \begin{pmatrix} \dot\gamma_{xx} & 0 & 0 \\ \dot\gamma_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \nabla n = \nabla T = \nabla \dot\gamma_{xa} = 0$$

$$\mathrm{USF}(a=y) : P_{xy} = -\sum_{k=0}^{\infty} \eta_k \left(\frac{\partial u_x}{\partial y}\right)^{2k+1}$$

$$\mathrm{ULF}(a=x) : P_{xx} = p - \frac{4}{3}\sum_{k=0}^{\infty} \eta'_k \left(\frac{\partial u_x}{\partial x}\right)^{k+1}$$

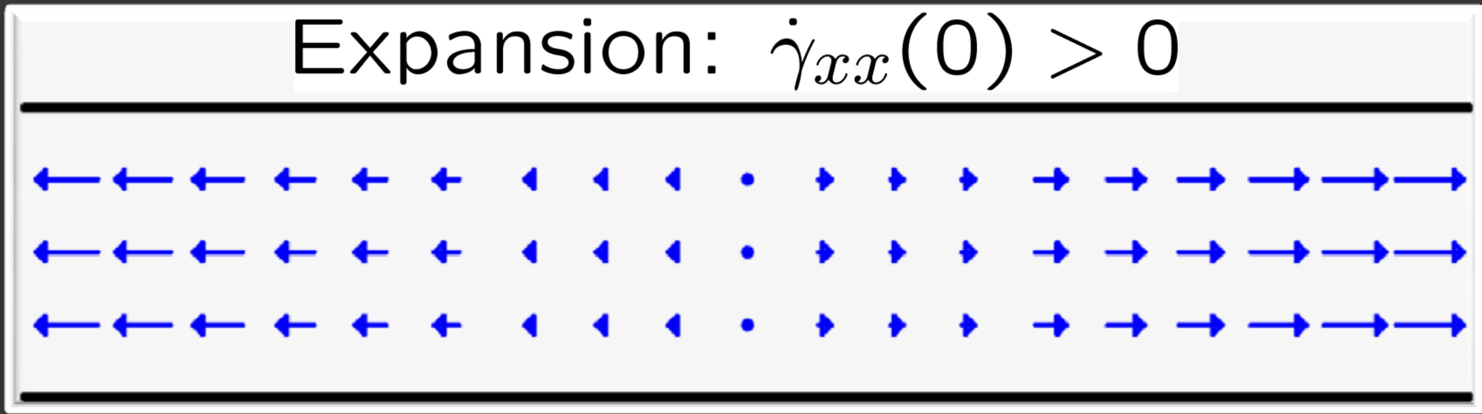
Uniform Shear Flow (USF)

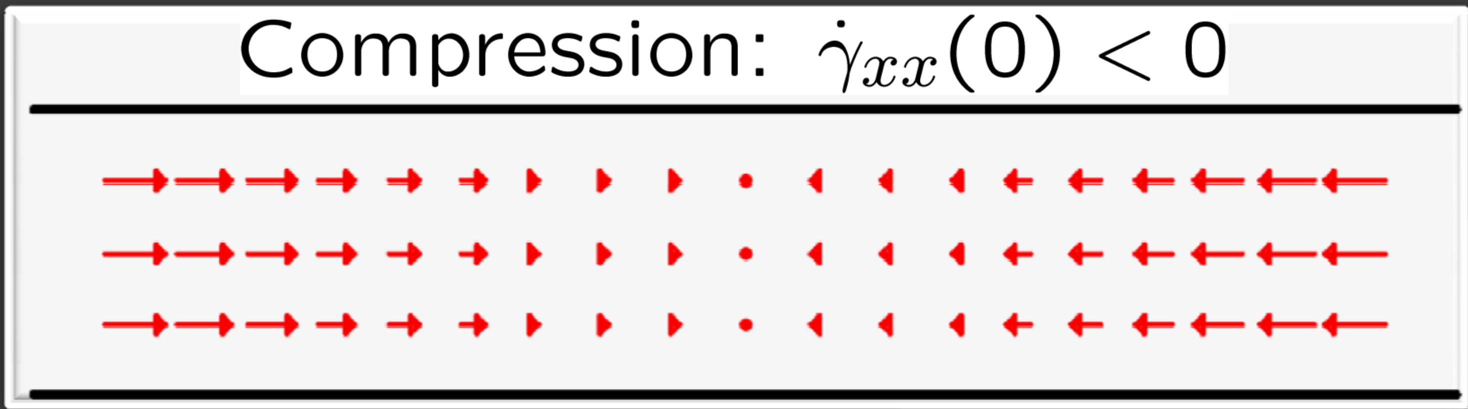
$$n(t) = n(0), \qquad \dot{\gamma}_{xy}(t) = \dot{\gamma}_{xy}(0)$$



Uniform Longitudinal Flow (ULF)

$$n(t) = \frac{n(0)}{1 + \dot{\gamma}_{xx}(0)t}, \qquad \dot{\gamma}_{xx}(t) = \frac{\dot{\gamma}_{xx}(0)}{1 + \dot{\gamma}_{xx}(0)t}$$





Uniformity parameter (or Knudsen number) in the USF and in the ULF

$$\left.\begin{array}{l} L_h \sim \dfrac{\sqrt{2T/m}}{|\dot\gamma_x a|} \\ \ell \sim \dfrac{\sqrt{2T/m}}{\nu} \end{array}\right\} \Rightarrow \mu = \frac{\dot\gamma_x a}{\nu} \propto \underbrace{\frac{1}{\sqrt{T}}}_{\text{Hard spheres}}$$

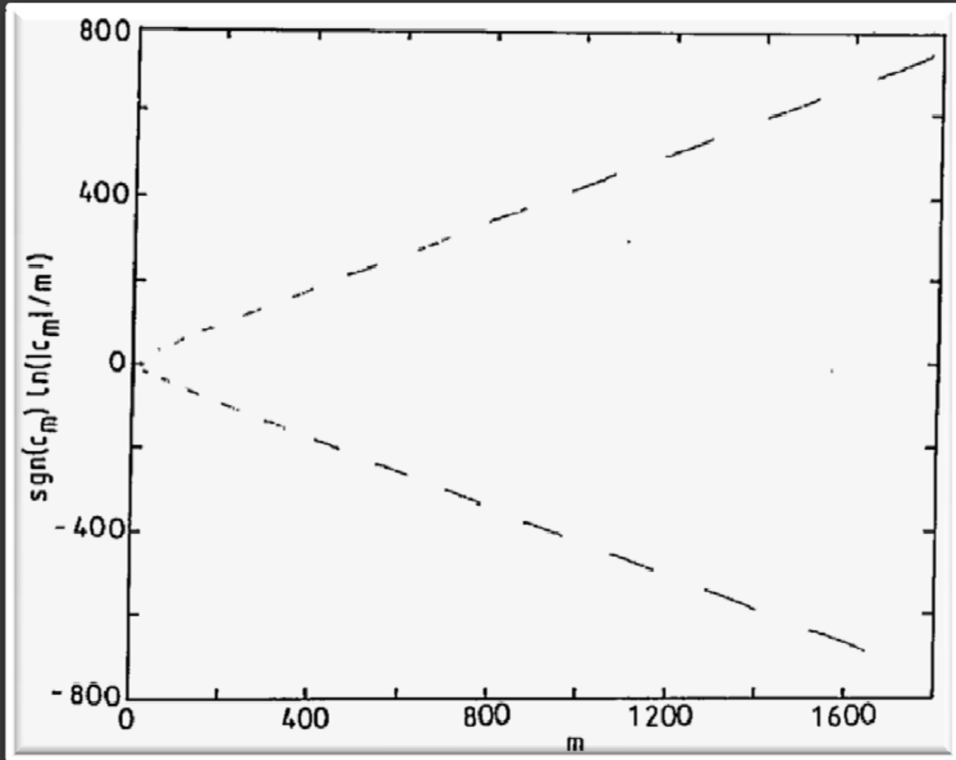
Non-Newtonian viscosity functions and scaled Chapman-Enskog expansions

$$\mathrm{USF}: \frac{P_{xy}}{p} = -\mu F_{xy}(\mu), \quad F_{xy}(\mu) = \sum_{k=0}^{\infty} c_k \mu^{2k}$$

$$\mathrm{ULF}: \frac{P_{xx}}{p} = 1 - \frac{4}{3}\mu F_{xx}(\mu), \quad F_{xx}(\mu) = \sum_{k=0}^{\infty} c'_k \mu^{k}$$

USF: The CE series diverges for *ordinary* gases

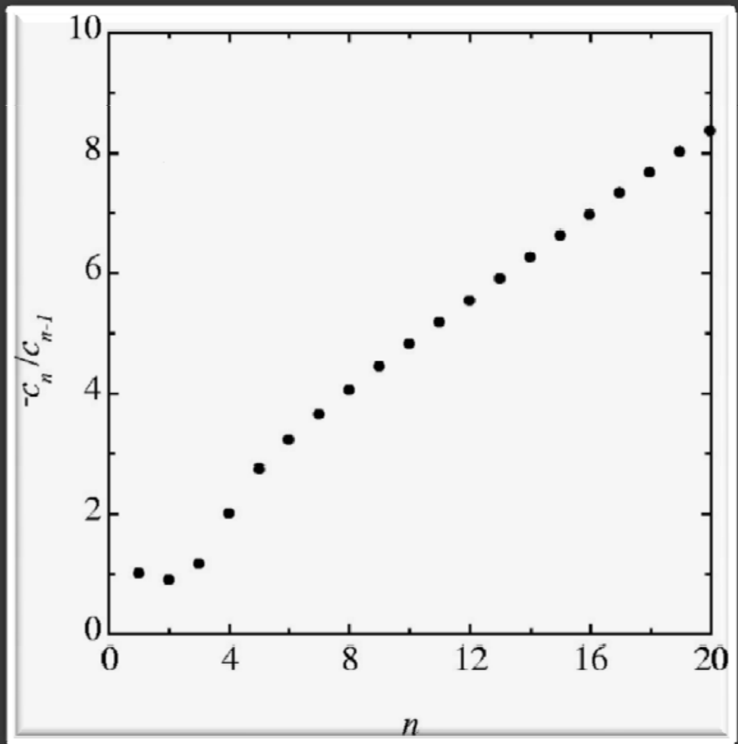
$$|c_k| \sim (2/3)^k \, k!$$



A.S., J. J. Brey, & J. W. Dufty, Phys. Rev. Lett. **56**, 1571 (1986)

ULF: Again, the CE series diverges for *ordinary* gases

$$|c'_k| \sim (1/3)^k \, k!$$



A.S., Phys. Rev. E **62**, 6597 (2000)

So far, we have restricted to *ordinary* gases (elastic collisions).

What happens in the case of granular gases (inelastic collisions)?

Taking into account that

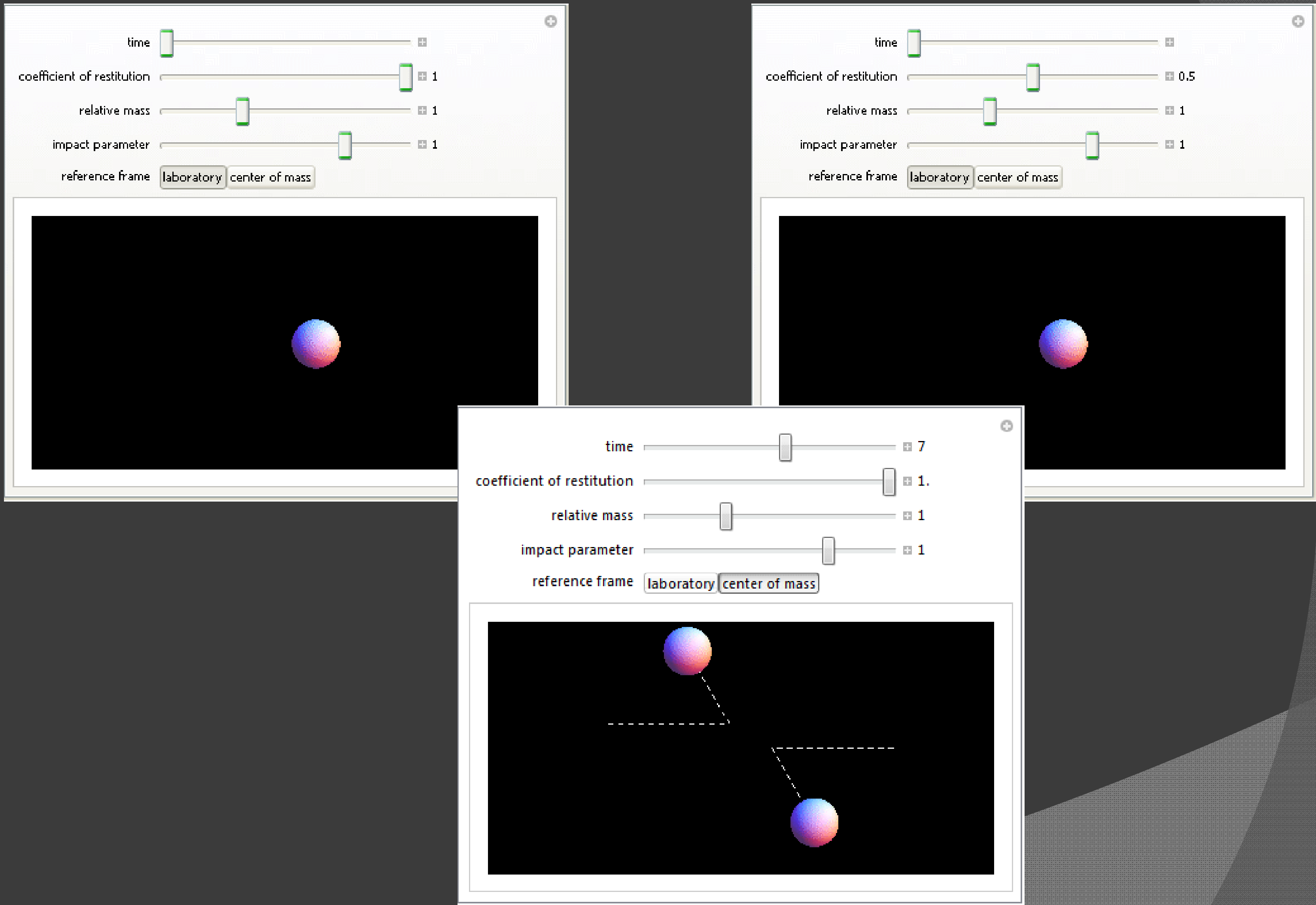
- The CE expansion diverges in the elastic case.
- Granular gases are inherently non-Newtonian (due to the coupling between inelasticity and gradients).
- Reasonable doubts about the applicability of hydrodynamics to granular gases.

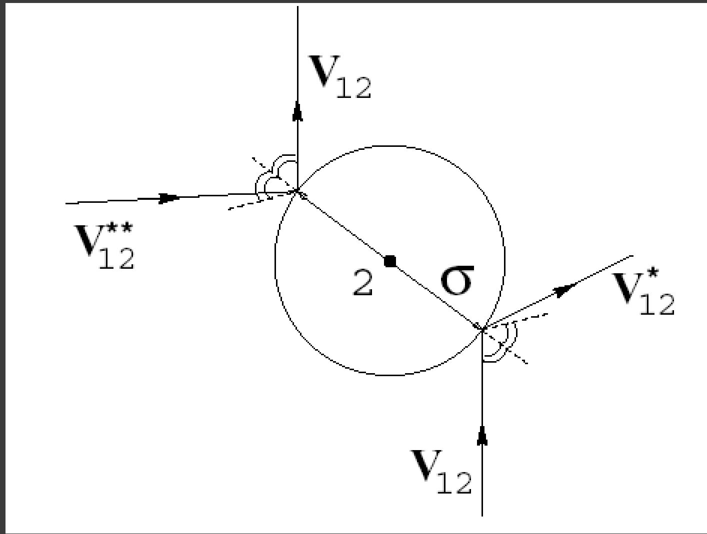
It seems natural to expect that the CE series is even more rapidly divergent for granular gases

Minimal model of a granular gas: A gas of (smooth) *inelastic* hard spheres

Several circles (Kandinsky, 1926)

http://demonstrations.wolfram.com/InelasticCollisionsOfTwoSpheres/





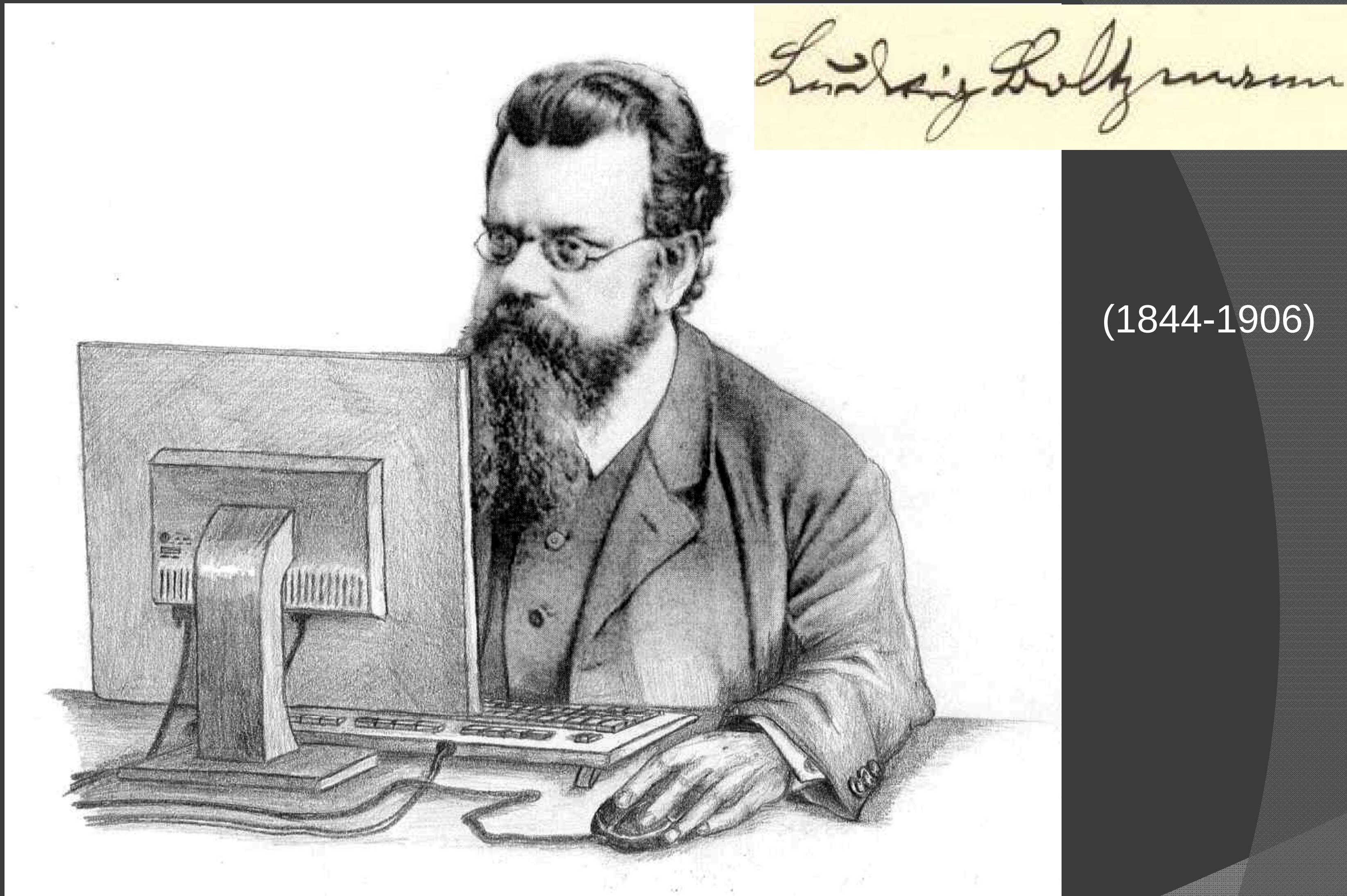
Collisions conserve momentum, but not kinetic energy:

$$\begin{aligned}\Delta E &= \frac{1}{2}m(v_1^{*2} + v_2^{*2} - v_1^2 - v_2^2) \\ &= -\frac{m}{2}(1-\alpha^2)(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})^2\end{aligned}$$

"Granular" temperature: $$T = \frac{m}{3}\langle(\mathbf{v}-\mathbf{u})^2\rangle, \quad \mathbf{u} = \langle\mathbf{v}\rangle$$

$$\left.\frac{\partial T}{\partial t}\right|_{\mathrm{coll}} = -\zeta T, \quad \zeta \sim 1-\alpha^2$$

"Cooling" rate



(1844-1906)

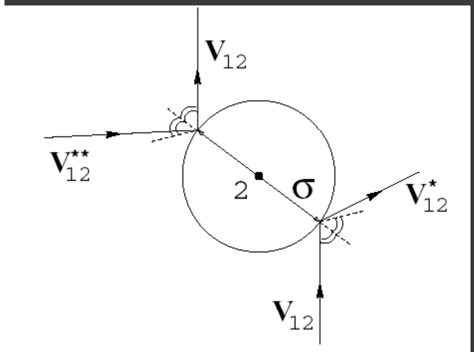
(Cartoon by Bernhard Reischl, University of Vienna)

Boltzmann equation (inelastic collisions)

$$\partial_t f + \mathbf{v}_1 \cdot \nabla f = J[f, f]$$

Collision operator

$$J[f,f] = \sigma^2 \int \mathrm{d}\mathbf{v}_2 \int \mathrm{d}\hat{\boldsymbol{\sigma}}\, \Theta(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}}) \times \left[\alpha^{-2} f(\mathbf{v}_1^{**}) f(\mathbf{v}_2^{**}) - f(\mathbf{v}_1) f(\mathbf{v}_2)\right]$$



$$\mathbf{v}_1^{**} = \mathbf{v}_1 - \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\hat{\sigma})\hat{\sigma}, \quad \mathbf{v}_2^{**} = \mathbf{v}_2 + \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\hat{\sigma})\hat{\sigma}$$

Energy balance equation

$$\partial_t T(t) = -\frac{2\dot{\gamma}_{xa}}{3n} P_{xa}(t) - \zeta(t) T(t)$$

Cooling rate

State	Viscous heating	Inelastic cooling	Stationary temperature
USF $(a = y)$	Yes	Yes	Yes
ULF $(a = x)$ $\dot{\gamma}_{xx} < 0$	Yes	Yes	Yes
ULF $(a = x)$ $\dot{\gamma}_{xx} > 0$	No	Yes	No $(T \to 0)$

Model kinetic equation (BGK-like)

$$(\partial_t + \mathbf{v}\cdot\nabla)f = -\nu(f - f_0) + \frac{\zeta}{2}\partial_{\mathbf{v}}\cdot[(\mathbf{v}-\mathbf{u})f]$$

$$J[f,f]$$

J. J. Brey, J. W. Dufty, & A. S., J. Stat. Phys. **97**, 281 (1999)

Moment equations for USF

$$\partial_t T = -\frac{2\dot{\gamma}_{xy}}{3n} P_{xy} - \zeta T$$

$$\begin{cases} \partial_t P_{xy} = -\dot{\gamma}_{xy} P_{yy} - (\nu + \zeta) P_{xy}, \\ \partial_t P_{yy} = \nu p - (\nu + \zeta) P_{yy}, \end{cases}$$

Stationary values of the reduced quantities:

$$\mu_s = \pm \sqrt{\frac{3\epsilon}{2}} (1 + \epsilon), \qquad F_{xy}(\mu_s) = \frac{1}{(1+\epsilon)^2}$$

$$\epsilon \equiv \frac{\zeta}{\nu} \propto 1 - \alpha$$

Moment equations for ULF

$$\partial_t T = -\frac{2\dot{\gamma}_{xx}}{3n} P_{xx} - \zeta T$$

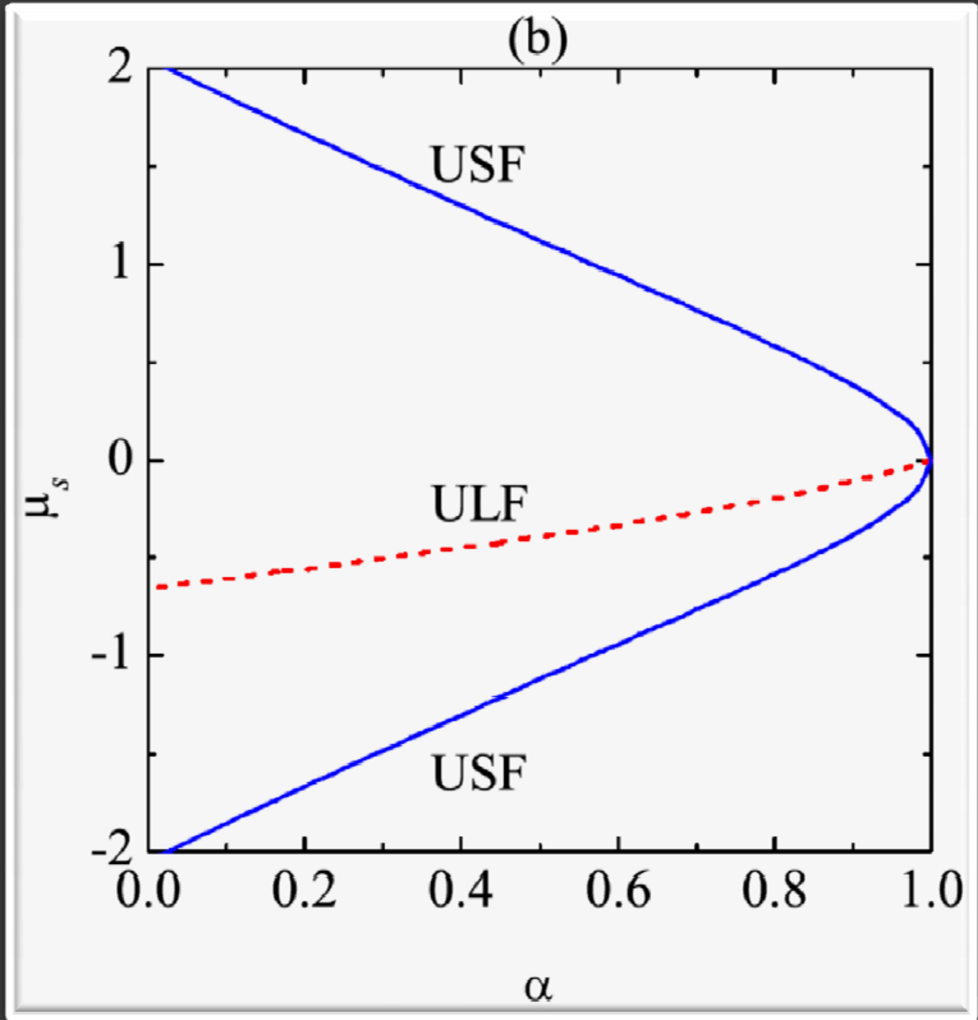
$$\partial_t P_{xx} = \nu p - (\nu + \zeta + 3\dot{\gamma}_{xx}) P_{xx}$$

Stationary values of the reduced quantities:

$$\mu_s = -\frac{3\epsilon(1+\epsilon)}{2(1+3\epsilon)}, \qquad F_{xx}(\mu_s) = \frac{1+3\epsilon}{(1+\epsilon)^2}$$

$$\epsilon \equiv \frac{\zeta}{\nu} \propto 1 - \alpha$$

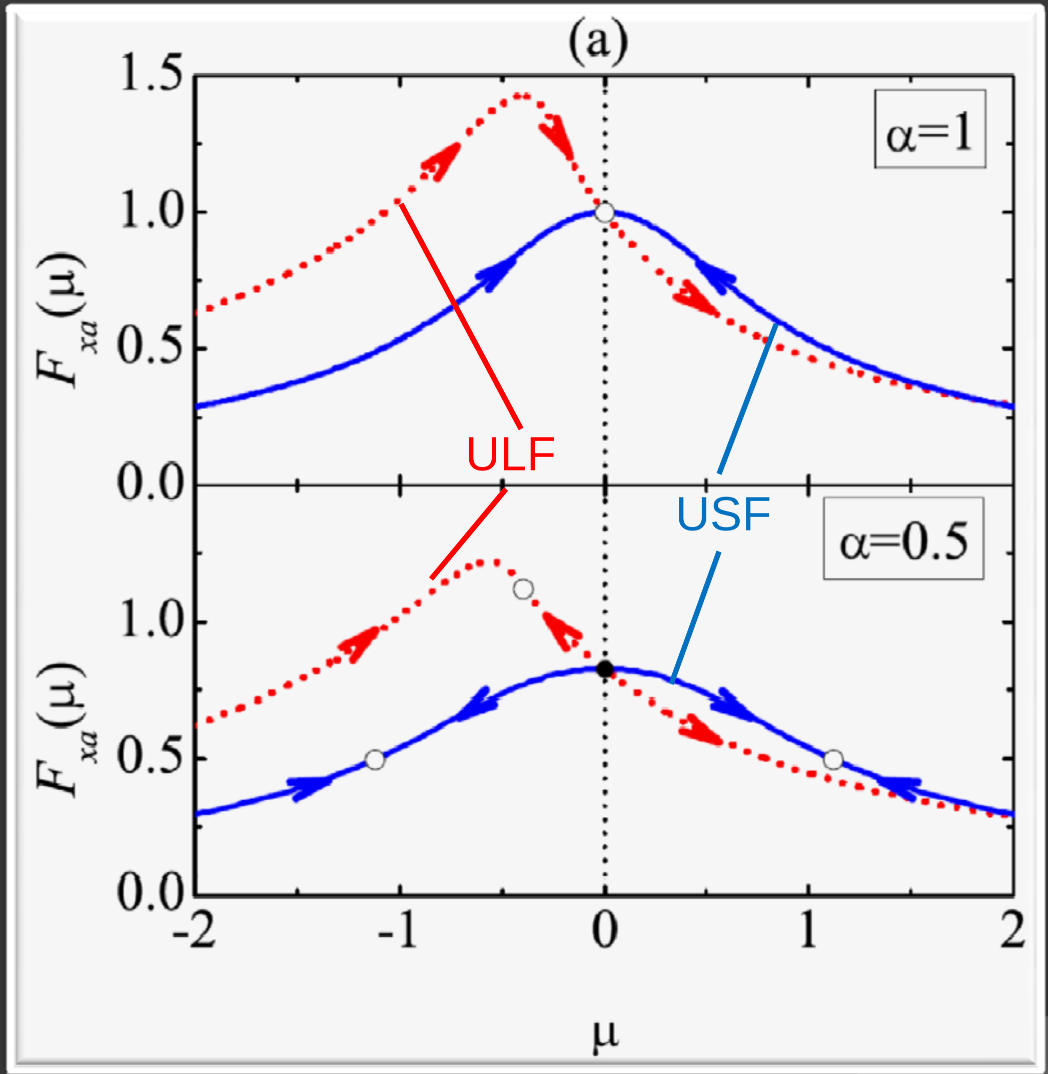
Stationary values



What about the whole viscosity function $F_{xa}(\mu)$?

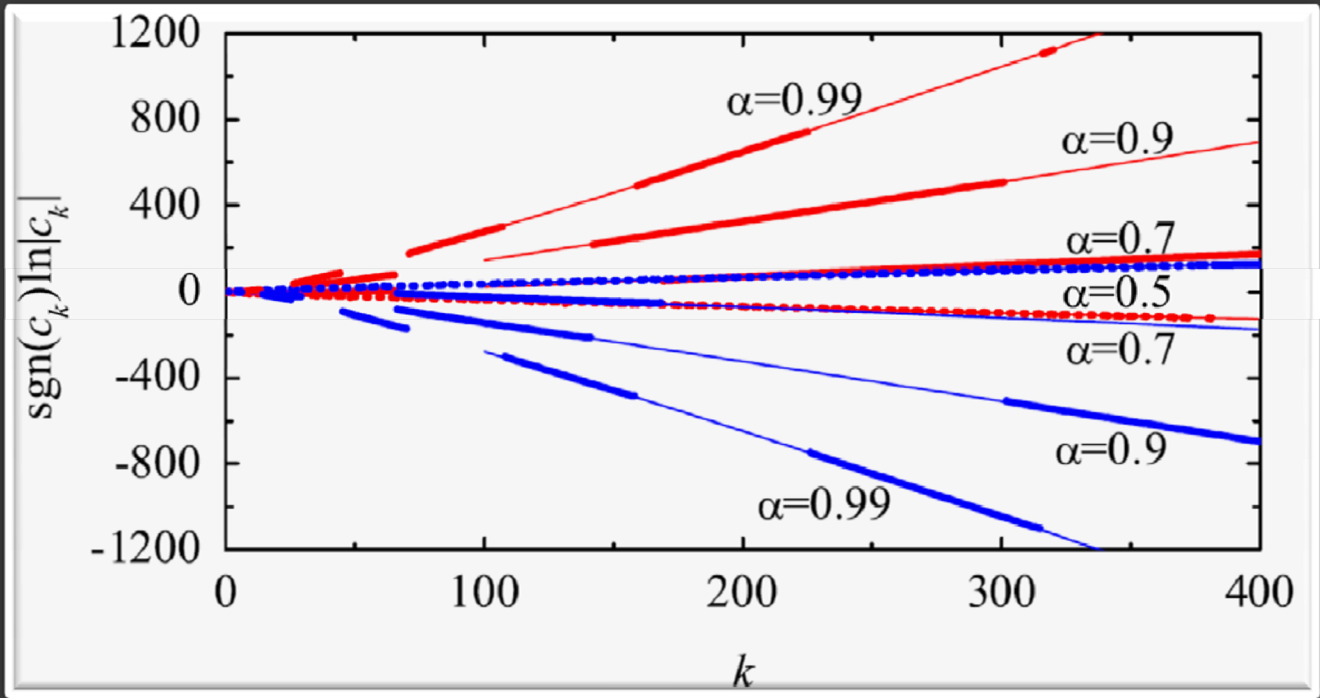
- By eliminating time in favor of $\mu(t)$ in the moment equations one gets a nonlinear 2nd-order ODE for $F_{xy}(\mu)$ and a nonlinear 1st-order ODE for $F_{xx}(\mu)$.
- They must be solved numerically.
- The ODEs yield recursion relations for c_k and c_k'.

Non-Newtonian viscosity



USF: Convergence of the CE expansion

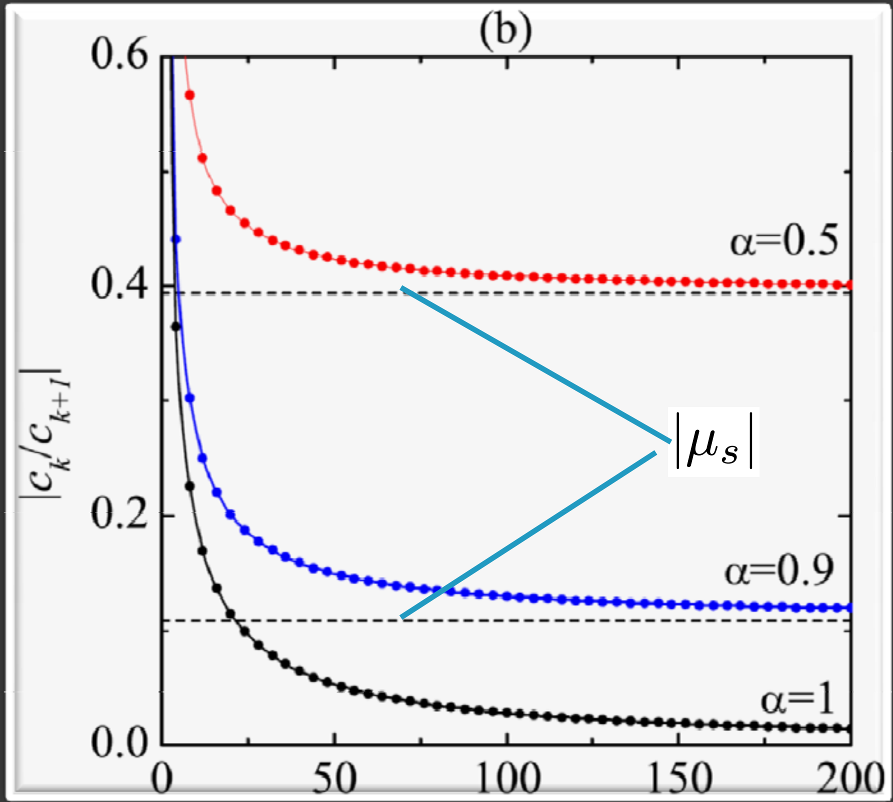
$$|c_k| \sim |\mu_s|^{-2k}$$



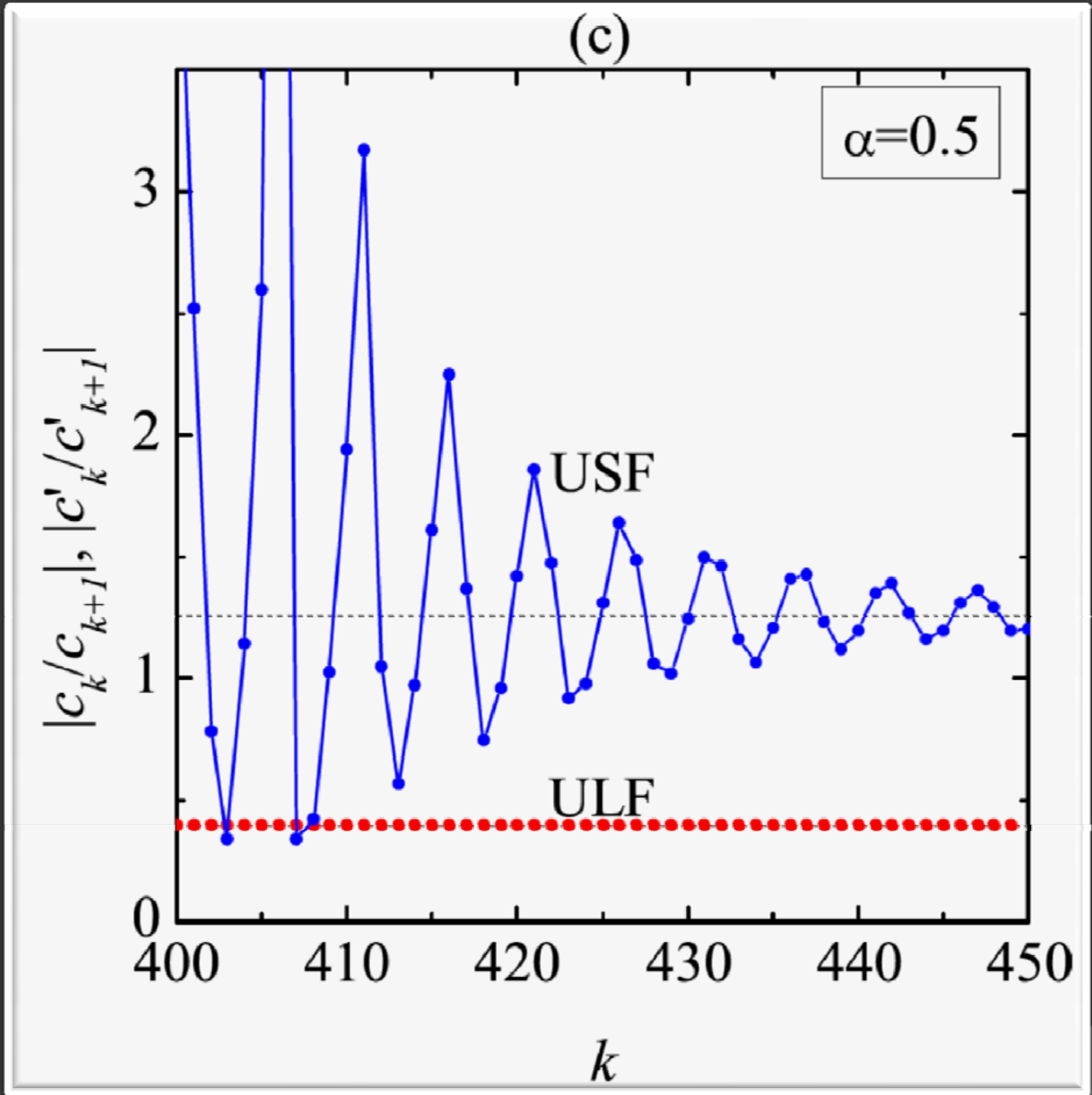
A.S., Phys. Rev. Lett. **100**, 078003 (2008)

ULF: Convergence of the CE expansion

$$|c_k'| \sim |\mu_s|^{-k}$$

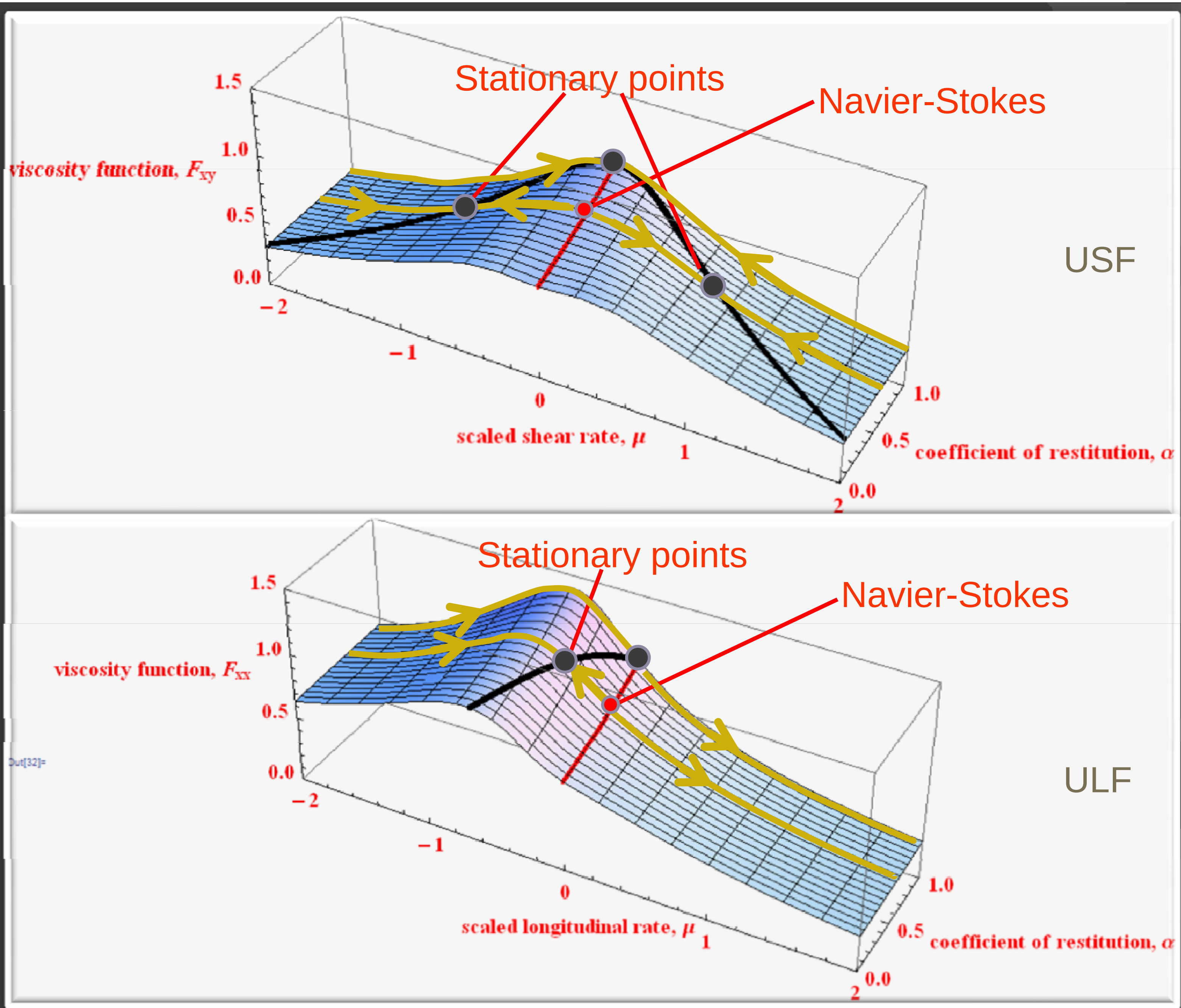


A.S., RGD 26 (AIP, 2009)


(c)
α=0.5
USF
ULF
3
2
1
0
400
410
420
430
440
450
k
$|c_k/c_{k+1}|, |c'_k/c'_{k+1}|$

Thus ...

- The Chapman-Enskog series diverges for *elastic* collisions.
- But it converges for *inelastic* collisions!
- In fact, the stronger the inelasticity, the larger the radius of convergence.
- Can this paradoxical result be understood by physical arguments?
- Yes! Just follow the arrow of time!


Stationary points
Navier-Stokes
viscosity function, F_{xy}
1.5
1.0
0.5
0.0
−2
−1
0
1
2
scaled shear rate, μ
coefficient of restitution, α
USF
Stationary points
Navier-Stokes
viscosity function, F_{xx}
scaled longitudinal rate, μ
coefficient of restitution, α
Out[32]=
ULF

Conclusions

- The reference homogeneous state (μ=0) is an *attractor* of the evolution of $\mu(t)$ for elastic collisions $\Rightarrow$ The CE expansion goes against the arrow of time $\Rightarrow$ The CE series diverges.
- The state μ=0 is a *repeller* of $\mu(t)$ for inelastic collisions $\Rightarrow$ The CE series converges.
- The convergence/divergence of the partial series of P_{xy} and P_{xx} is independent of whether actually the gas is or not in the USF or in the ULF.
- There exists a close relationship between the inherent non-Newtonian nature of granular steady states and the convergence of the CE expansion.

Merci beaucoup par votre attention!