Shear viscosity of a granular fluid

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OUTLINE

I. Undriven Uniform Shear Flow

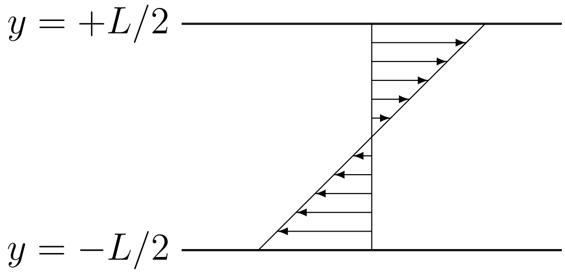
- One of the simplest nonequilibrium states: **Uniform Shear Flow** (USF)

- At a *macroscopic* level, the USF is characterized by

  $u_x = ay$, a =constant shear rate

  n =const

  $\nabla T = 0$



- In the local Lagrangian frame, the velocity distribution function becomes *uniform*:

$$f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{V}, t), \quad \mathbf{V} \equiv \mathbf{v} - \mathbf{u}$$

Molecular fluid (elastic collisions)

- Energy balance equation ⇒ Viscous heating:

$$\partial_t T = \underbrace{-\frac{2}{3n} a P_{xy}}_{\text{viscous heating}} , \quad P_{ij} = \text{pressure tensor}$$

- Low-density gas of hard spheres [Gómez-Ordóñez *et al.*, PRA **39**, 3038 (1989)]:

- Collision frequency grows with time: $\nu(t) \propto [T(t)]^{1/2}$ (hard spheres)

- Reduced shear rate: $a^*(t) \equiv \frac{a}{\nu(t)} \stackrel{t\to\infty}{\longrightarrow} 0$

- This is an efficient way of measuring the *Navier-Stokes* shear viscosity $\eta_0(n, T)$, as first proposed by Naitoh & Ono (1979):

$$-a^{-1}P_{xy}(t) \to \eta_0(n, T(t))$$

$$\Rightarrow -\frac{\nu(t)}{a}\frac{P_{xy}(t)}{nT(t)} \stackrel{t\to\infty}{\longrightarrow} \frac{\nu(t)}{nT(t)}\eta_0(n, T(t)) = \eta_0^*(\phi)$$

- For instance, at a packing fraction $\phi \equiv (\pi/6)n\sigma^3 = 0.42$ [Montanero & Santos, PRE **54**, 438 (1996)],

Granular fluid (inelastic collisions)

- Now, the inelasticity of collisions provides an energy sink:

$$\partial_t T = \underbrace{-\frac{2}{3n} a P_{xy}}_{\text{viscous heating}} + \underbrace{(-\zeta T)}_{\text{inelastic cooling}} \quad , \quad \zeta = \text{cooling rate} \propto T^{1/2}(1-\alpha^2)$$

- A steady state is eventually reached in which the viscous heating is exactly balanced by collisional cooling effects:

$$-\frac{P_{xy}}{nT} \to \frac{3\zeta}{2a}$$

- Dimensional analysis:

$$-\frac{P_{xy}}{nT} = F(\phi,\alpha) \Rightarrow \begin{cases} T = \theta(\phi,\alpha) m\sigma^2 a^2 \propto a^2 \\ -P_{xy} = \phi^{-1}\tau_{xy}(\phi,\alpha) mn\sigma^2 a^2 \propto a^2 \end{cases}$$

Highly non-Newtonian behavior!

- Functions $\theta(\phi,\alpha)$ and $\tau_{xy}(\phi,\alpha)$ [Montanero *et al.* JFM **389**, 391 (1999)]:

II. Driven Uniform Shear Flow

- Thus, in the (undriven) USF for granular fluids, $-P_{xy} \propto a^2$.
- Is it possible to "frustrate" the cooling effects so that viscous heating dominates and $-P_{xy} \propto a$ for long times, as in the case of molecular fluids?
- If so, one could identify a (linear) shear viscosity as

$$-a^{-1}P_{xy}(t) \to \eta(\alpha; n, T(t))$$

$$\Rightarrow -\frac{\nu(t)}{a}\frac{P_{xy}(t)}{nT(t)} \xrightarrow{t\to\infty} \frac{\nu(t)}{nT(t)}\eta(\alpha; n, T(t)) = \eta^*(\alpha, \phi)$$

- How different is $\eta^*(\alpha, \phi)$ from $\eta^*(\alpha = 1, \phi) \equiv \eta_0^*(\phi)$?
- To answer these questions, let us assume that the granular fluid is excited by an external energy source that exactly compensates for the collisional loss:

$$\partial_t T = \underbrace{-\frac{2}{3n}aP_{xy}}_{\text{viscous heating}} + \underbrace{(-\zeta T)}_{\text{inelastic cooling}} + \underbrace{\zeta T}_{\text{external source}}$$

- The simplest choice for such an excitation is an "anti-drag" force of the form

$$\mathbf{F}_{\text{exc}} = \frac{1}{2}\zeta(\mathbf{v} - \mathbf{u})$$

- In the absence of shear ($a = 0$, HCS), $\mathbf{F}_{\text{exc}}$ *does not* affect the dynamics of the system since it is equivalent to a rescaling of the velocities of the particles.

III. Enskog Theory

- The Enskog equation for inelastic hard spheres under USF is

$$\partial_t f + \underbrace{(-aV_y \partial_{V_x} f)}_{\text{inertial force}} + \underbrace{\frac{\zeta}{2}\partial_{\mathbf{V}} \cdot \mathbf{V} f}_{\text{external excitation}} = \underbrace{J^E[\mathbf{V}|f]}_{\text{inelastic collisions}}$$

where

$$J^E[\mathbf{V}|f] = \sigma^2 \chi(n) \int d\mathbf{V}_1 \int d\widehat{\boldsymbol{\sigma}}\, \Theta(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})\, (\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})[\alpha^{-2} f(\mathbf{V}', t) f(\mathbf{V}_1', t) - f(\mathbf{V}, t) f(\mathbf{V}_1, t)]$$

$$\mathbf{g} = \mathbf{V} - \mathbf{V}_1 - \sigma a \widehat{\sigma}_y \widehat{\mathbf{x}},$$

$$\mathbf{V}' = \mathbf{V} - \frac{1+\alpha^{-1}}{2}(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})\widehat{\boldsymbol{\sigma}}, \quad \mathbf{V}_1' = \mathbf{V}_1 + \frac{1+\alpha^{-1}}{2}(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})\widehat{\boldsymbol{\sigma}} + 2\sigma a \widehat{\sigma}_y \widehat{\mathbf{x}}$$

- Our aim is to get $\eta^*(\alpha, \phi)$ by a two-fold route:

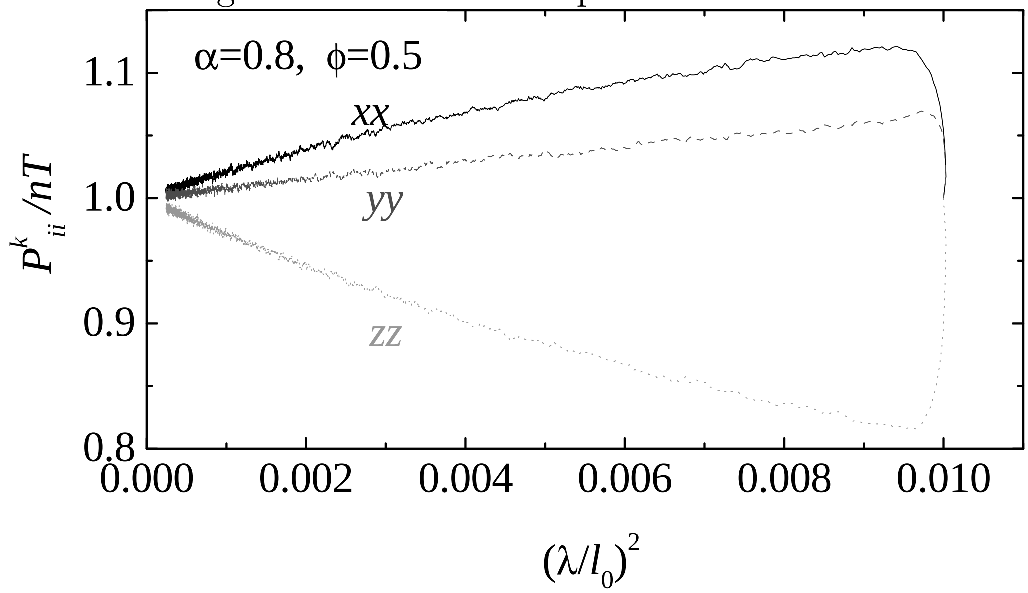
  1. Monte Carlo simulations by a variant of the DSMC method [Montanero & Santos, PF **9**, 2057 (1997)].

  2. Perturbation analysis around the HCS + Sonine approximation.

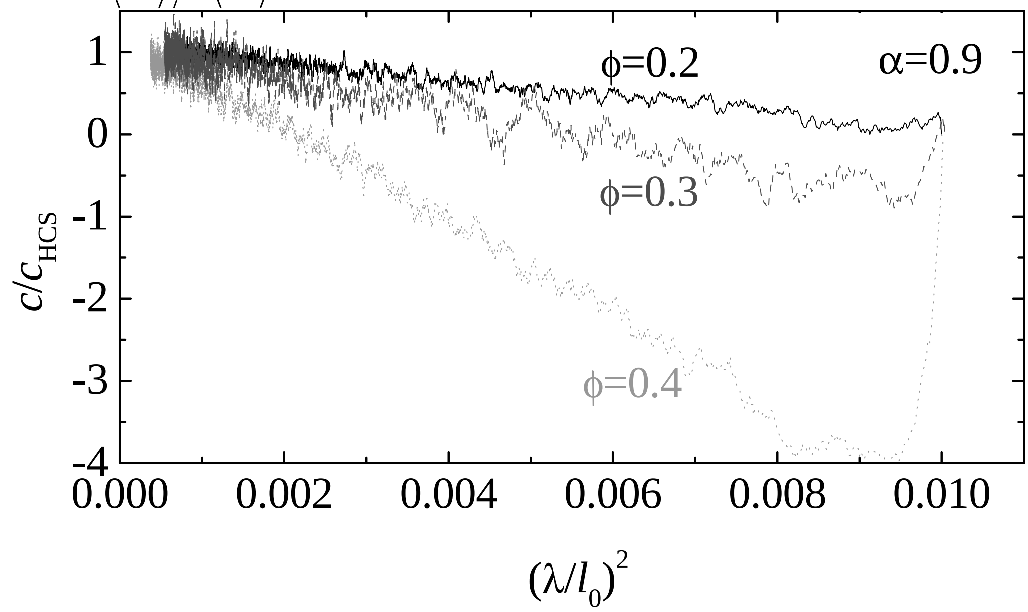
Monte Carlo simulations

- Time is monitored by $\lambda/l_0 \propto a^*$, where $\lambda = [\sqrt{2}\pi n\sigma^2\chi(n)]^{-1}$ is the mean free path and $l_0 = \sqrt{2T/m}/a$ is the characteristic hydrodynamic length, which increases (almost linearly) with time.

- Kinetic part of the diagonal elements of the pressure tensor:



- Cumulant $c = 3\langle v^4\rangle/5\langle v^2\rangle^2 - 1$:



• Marginal distribution functions:

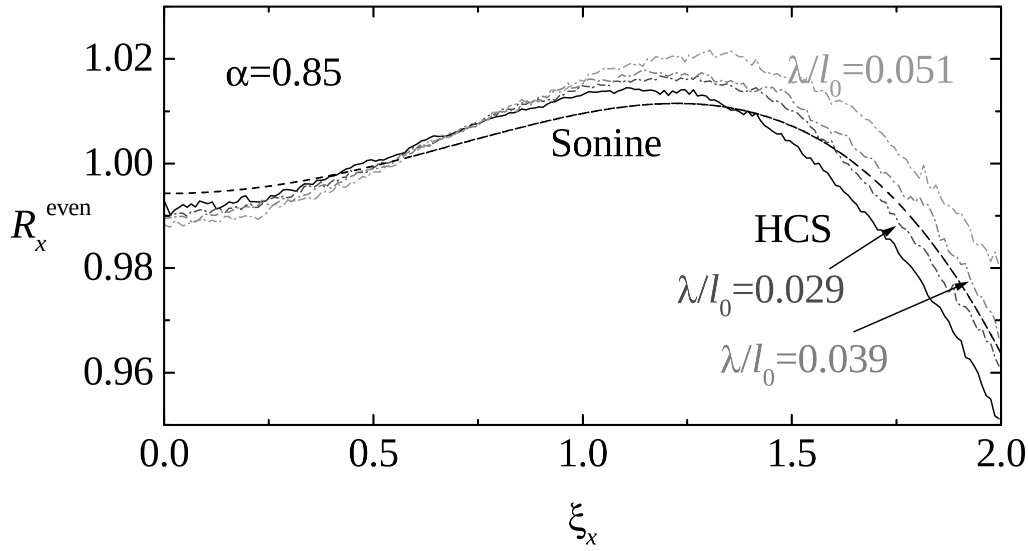
$$\varphi_x(V_x,t)=\int_0^\infty dV_y\int_{-\infty}^\infty dV_z f(\mathbf{V},t)$$

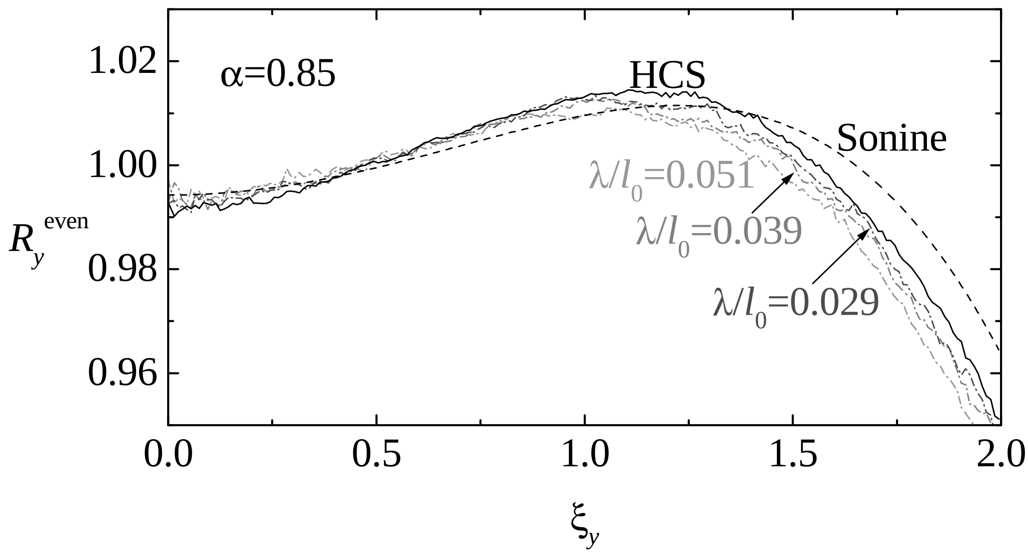
• Even and odd parts:

$$\varphi_x^{\rm even}(V_x,t)=\frac{1}{2}\left[\varphi_x(V_x,t)+\varphi_x(-V_x,t)\right],\quad \varphi_x^{\rm odd}(V_x,t)=\frac{1}{2}\left[\varphi_x(V_x,t)-\varphi_x(-V_x,t)\right]$$

• Normalized even distribution:

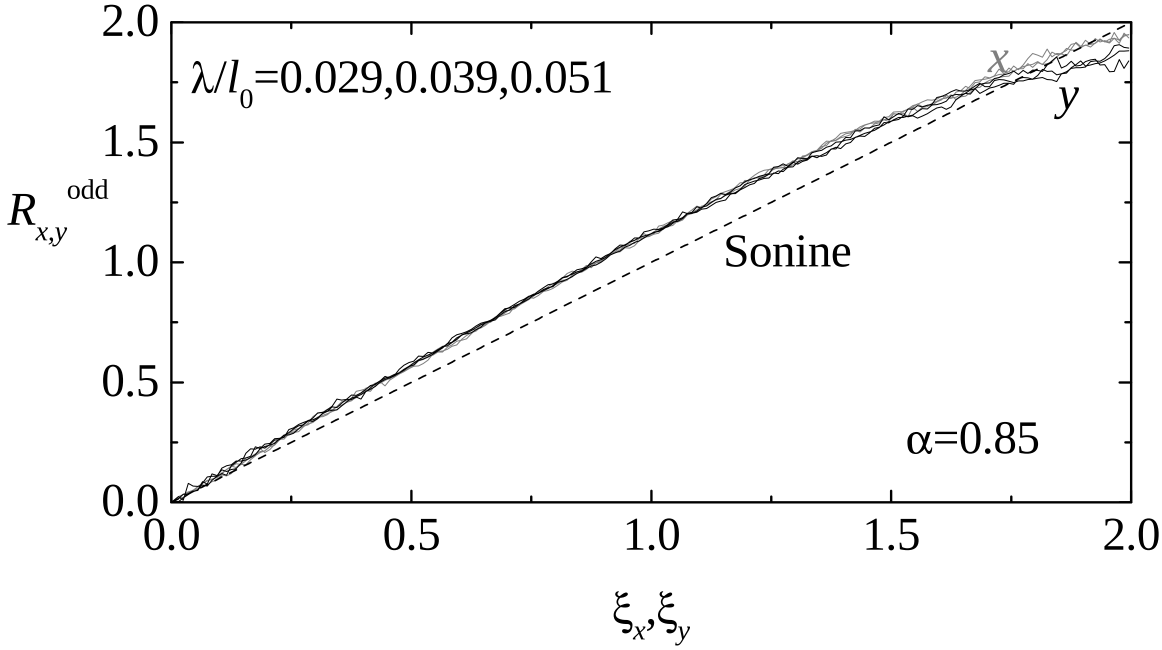
$$R_x^{\rm even}(\xi_x)=\frac{\varphi_x^{\rm even}(V_x,t)}{\varphi_x^{\rm MB}(V_x,t)},\quad \xi_x=V_x/\sqrt{2T/m}$$





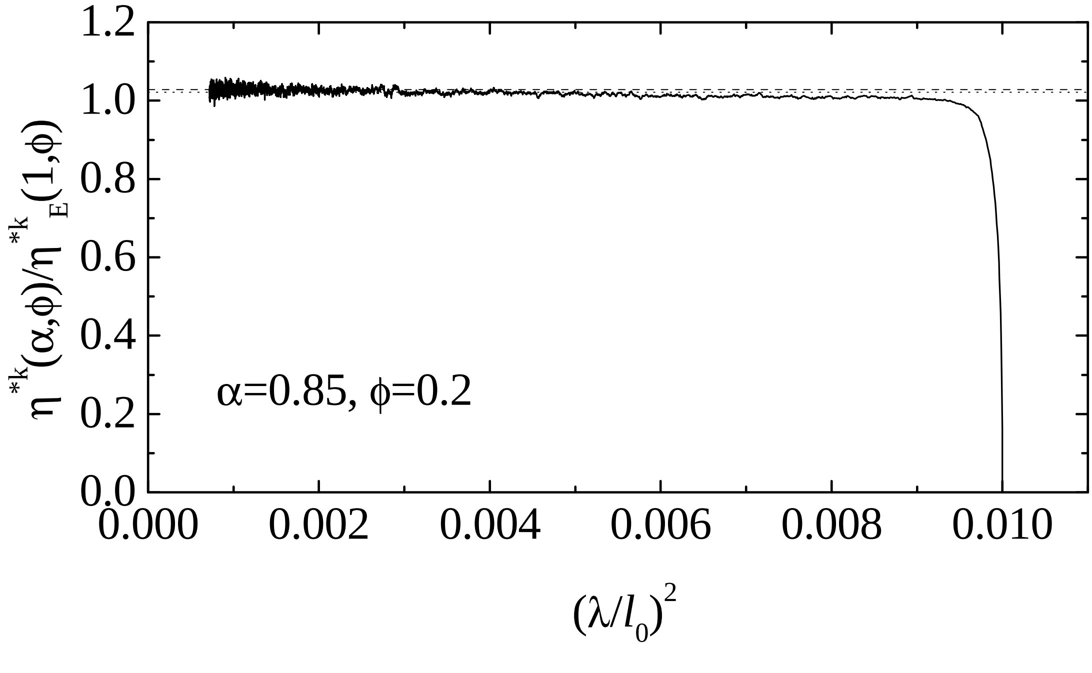
• Normalized odd distribution:

$$R_x^{\mathrm{odd}}(\xi_x) = \frac{\varphi_x^{\mathrm{odd}}(V_x,t)}{\varphi_x^{\mathrm{MB}}(V_x,t)} \frac{\sqrt{\pi} n T(t)}{2P_{xy}^k(t)}$$



• Time evolution of the (kinetic part of the) viscosity:

$$\eta^{*k} = -\frac{\nu(t)}{a} \frac{P_{xy}^k(t)}{nT(t)}$$



Perturbation analysis

- Kinetic equation:

$$\partial_t f + (-aV_y \partial_{V_x} f) + \frac{\zeta}{2} \partial_{\mathbf{V}} \cdot \mathbf{V} f = J^E[\mathbf{V}|f]$$

- Perturbation expansion (*à la* Chapman-Enskog) in powers of the shear rate:

$$f(\mathbf{V}) = \underbrace{f_0(\mathbf{V})}_{\text{HCS}} + \underbrace{f_1(\mathbf{V})}_{\mathcal{O}(a)} + \mathcal{O}(a^2)$$

$$J^E[\mathbf{V}|f] = \underbrace{J_0^E[\mathbf{V}|f_0]}_{\text{HCS}} + \underbrace{J_1^E[\mathbf{V}|f_0] - \mathcal{L} f_1(\mathbf{V})}_{\mathcal{O}(a)} + \mathcal{O}(a^2)$$

$$\zeta = \underbrace{\zeta_0}_{\text{HCS}} + \mathcal{O}(a^2)$$

$$\partial_t = \mathcal{O}(a^2)$$

- Zeroth order:

$$\frac{\zeta_0}{2} \partial_{\mathbf{V}} \cdot \mathbf{V} f_0 = J_0^E[\mathbf{V}|f_0]$$

- First order:

$$aV_y \partial_{V_x} f_0 + J_1^E[\mathbf{V}|f_0] = \left(\mathcal{L} + \frac{\zeta_0}{2} \partial_{\mathbf{V}} \cdot \mathbf{V}\right) f_1$$

• Sonine approximation:

$$f_0(\mathbf{V}) \to f_{\rm MB}(\mathbf{V})\left[1+c(\alpha)S_2(\xi^2)\right], \quad S_2(x)=\frac{1}{2}x^2-\frac{5}{2}x+\frac{15}{8}$$

$$f_1(\mathbf{V}) \to -\frac{ma\eta^k}{nT^2}f_{\rm MB}(\mathbf{V})V_xV_y$$

• This allows us to get explicit expressions for the transport coefficients $\eta^{*k}(\alpha,\phi)$ and $\eta^*(\alpha,\phi)$.

• The theory predicts that the shear viscosity of the inelastic system is larger than that of the elastic system at the same density, $\eta^*(\alpha,\phi)>\eta^*(1,\phi)$, *if* the packing fraction is smaller than a threshold value, $\phi<\phi_0(\alpha)$, while the opposite happens if $\phi>\phi_0(\alpha)$.

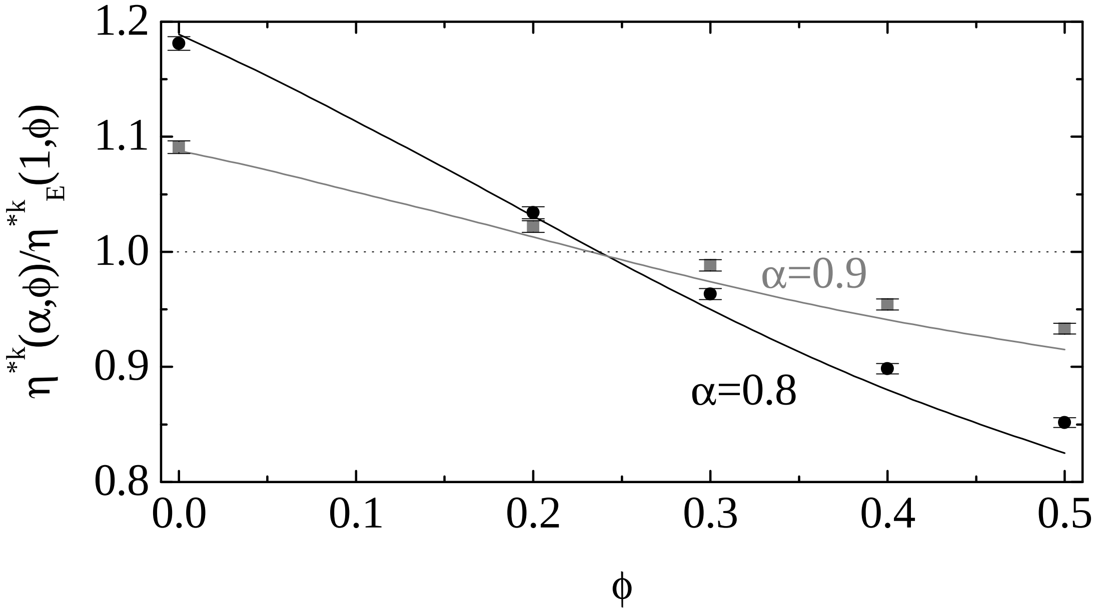
Similar threshold values $\phi_0^k(\alpha)$ and $\phi_0^c(\alpha)$ exist for the kinetic and collisional parts of the shear viscosity.

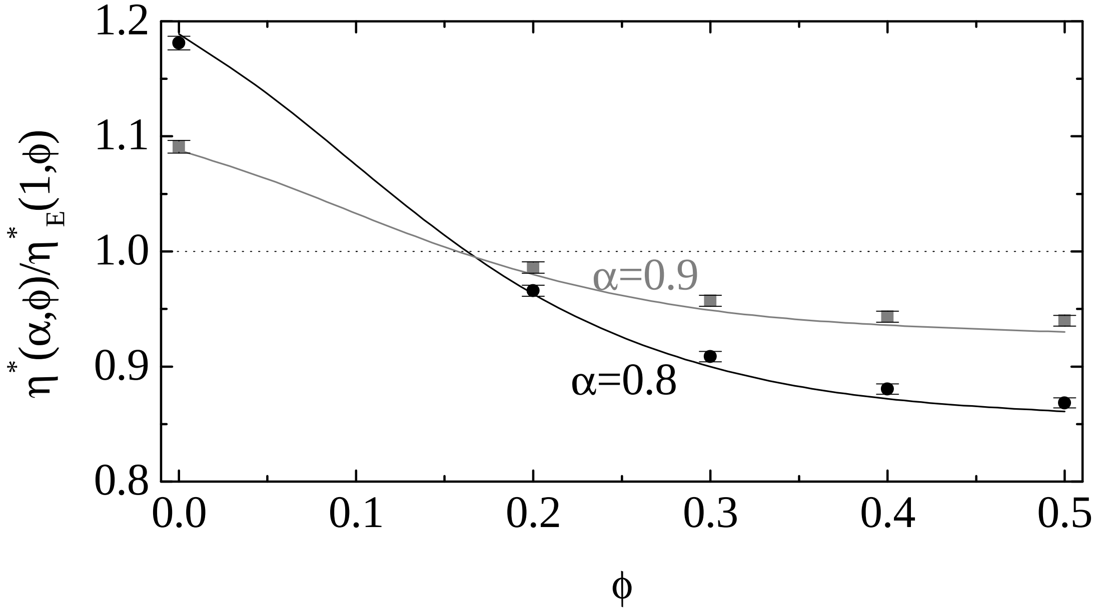
• In the range $0.8\le\alpha\le 1$ the threshold values are practically independent of the coefficient of restitution:

$$\phi_0(\alpha)\simeq 0.16, \quad \phi_0^k(\alpha)\simeq 0.23, \quad \phi_0^c(\alpha)\simeq 0.05$$

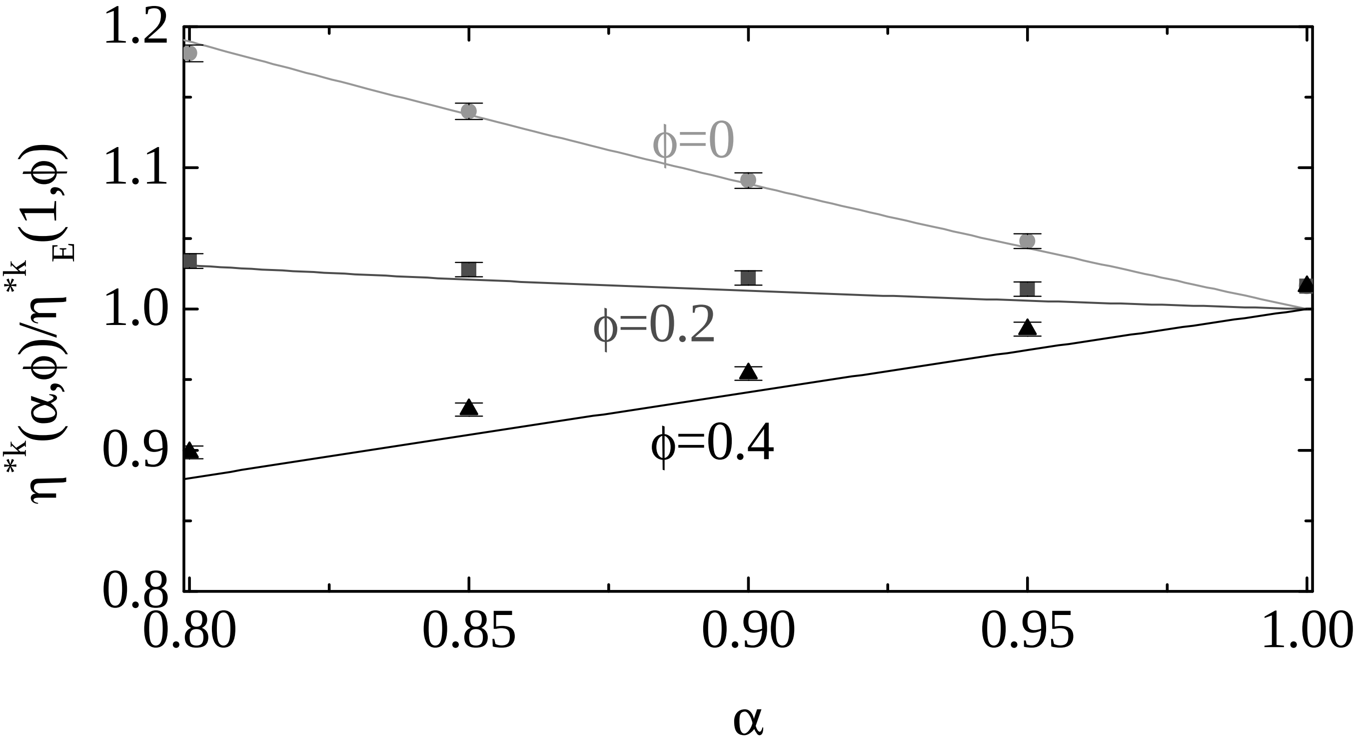
IV. Results

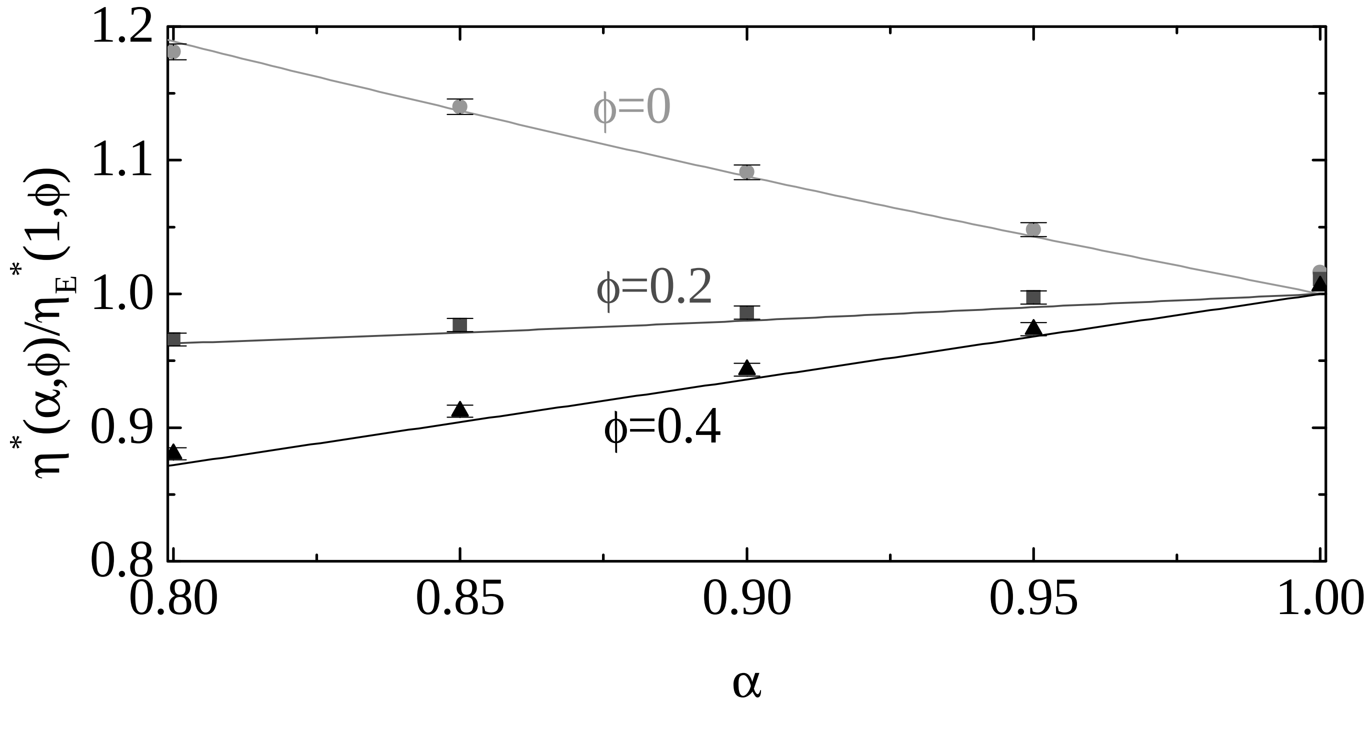
- Density dependence:





• Coefficient of restitution dependence:





V. Conclusions

- A driven system of inelastic hard spheres under USF reaches for long times a hydrodynamic regime in which the shear stress is proportional to the shear rate, $P_{xy} = -\eta a$. The proportionality constant defines a shear viscosity coefficient $\eta(\alpha; n, T)$ as a material function of the coefficient of restitution, density, and temperature.

- Comparison between Monte Carlo simulation data and theoretical results obtained from a perturbation analysis (plus a Sonine approximation) shows an excellent agreement.

- The granular fluid is less (more) viscous than the corresponding molecular one if the packing fraction is larger (smaller) than about 16%.

- The same type of excitation mechanism is easy to implement in molecular dynamics simulations. This would be an efficient way of measuring the linear shear viscosity $\eta(\alpha; n, T)$ and compare it with the results obtained from the Enskog theory.

- The coefficient $\eta(\alpha; n, T)$ represents the linear shear viscosity of an excited granular fluid under USF. Does it coincide with the *Navier-Stokes* shear viscosity, $\eta_{\rm NS}(\alpha; n, T)$, characterizing the response of the system to a weak spontaneous inhomogeneity in the velocity field?

- In the latter case, a Chapman-Enskog expansion [Garzó & Dufty, PRE **59**, 5895 (1999)] of the form $f = f_0 + f_{\rm NS} + \cdots$ leads to

$$aV_y \partial_{V_x} f_0 + J_1^E[\mathbf{V}|f_0] = \left(\mathcal{L} + \frac{\zeta_0}{2}\partial_{\mathbf{V}} \cdot \mathbf{V} + \frac{\zeta_0}{2}\right) f_{\rm NS}$$

while in our problem we had

$$aV_y \partial_{V_x} f_0 + J_1^E[\mathbf{V}|f_0] = \left(\mathcal{L} + \frac{\zeta_0}{2}\partial_{\mathbf{V}} \cdot \mathbf{V}\right) f_1$$

Thus, $f_1 \neq f_{\rm NS}$ and, consequently, $\eta \neq \eta_{\rm NS}$.

- In fact, in the Sonine approximation,

$$\frac{1}{\eta_{\rm NS}^{*k}} = \frac{1}{\eta^{*k}} + \frac{5}{24}(1-\alpha)^2 \chi(\phi) \frac{1 + \frac{3}{16}c(\alpha)}{1 - \frac{2}{5}(1+\alpha)(1-3\alpha)\phi\chi(\phi)}$$

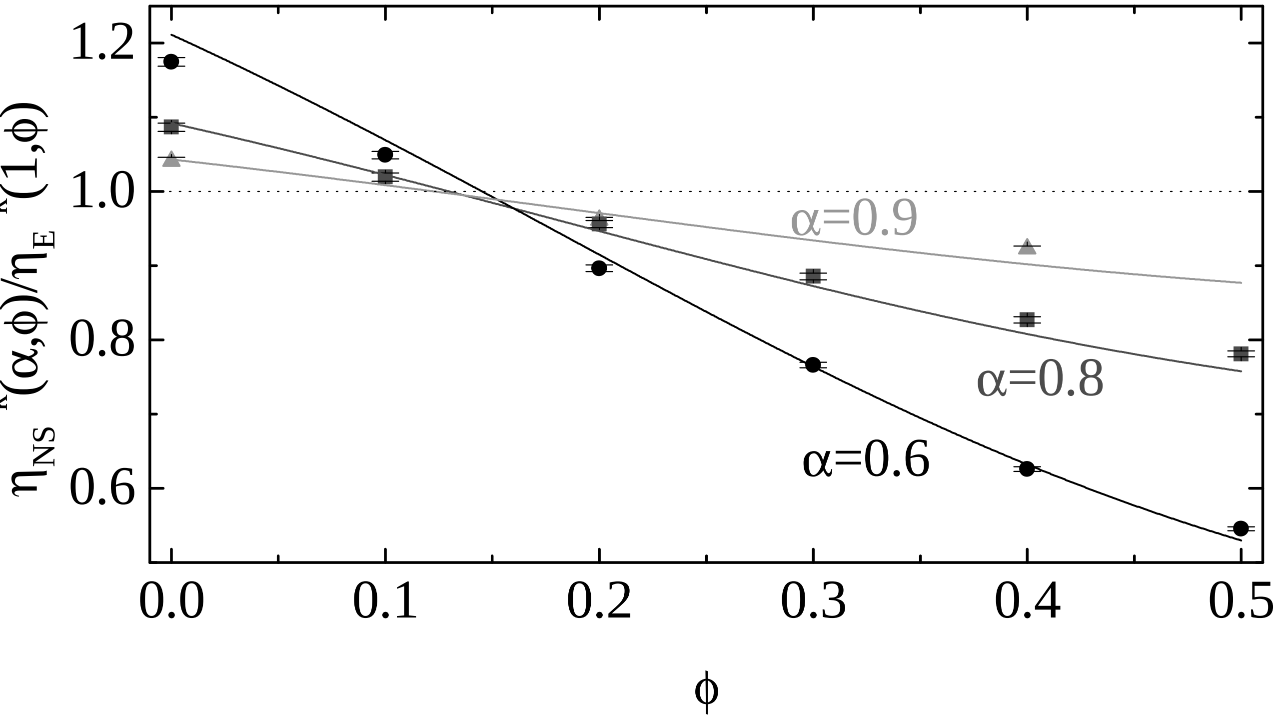
- Is it possible to "retouch" the driven USF problem so that the coefficient $\eta_{\rm NS}$, rather than η, can be measured in simulations?

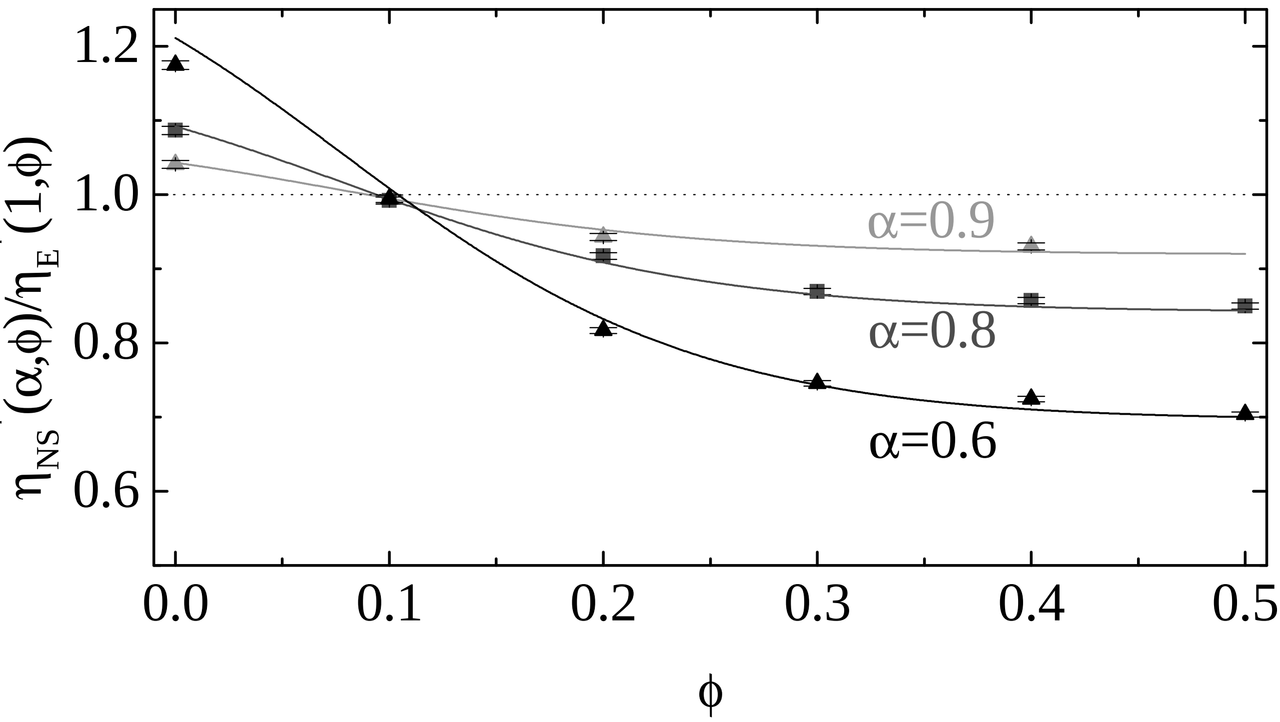
- The simplest possibility is

$$\partial_t f + \underbrace{(-aV_y \partial_{V_x} f)}_{\text{inertial force}} + \underbrace{\frac{\zeta}{2}\partial_{\mathbf{V}} \cdot \mathbf{V} f}_{\text{external excitation}} = \underbrace{J^E[\mathbf{V}|f]}_{\text{inelastic collisions}} - \underbrace{\frac{\zeta}{2}(f - f_0)}_{\text{BGK-like term}}$$

- In the simulations the new term is implemented by randomly choosing a fraction of particles $\zeta \delta t/2$ in each timestep δt and replacing the velocities of those particles by random velocities drawn from the distribution f_0.

• Density dependence:





• In this case, $\phi_0(\alpha) \simeq 0.10, \phi_0^k(\alpha) \simeq 0.13, \phi_0^c(\alpha) = 0$

• Coefficient of restitution dependence: