LONGITUDINAL VISCOUS FLOW IN GRANULAR GASES

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Outline

- A granular gas modeled as inelastic hard spheres.
- Longitudinal flow.
- Generalized (nonlinear) viscosity function.
- Does the Chapman-Enskog expansion of the generalized viscosity converge?
- Conclusions.

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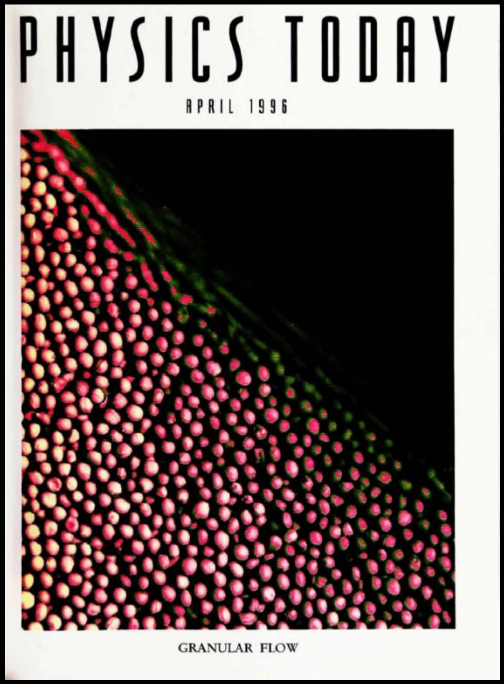
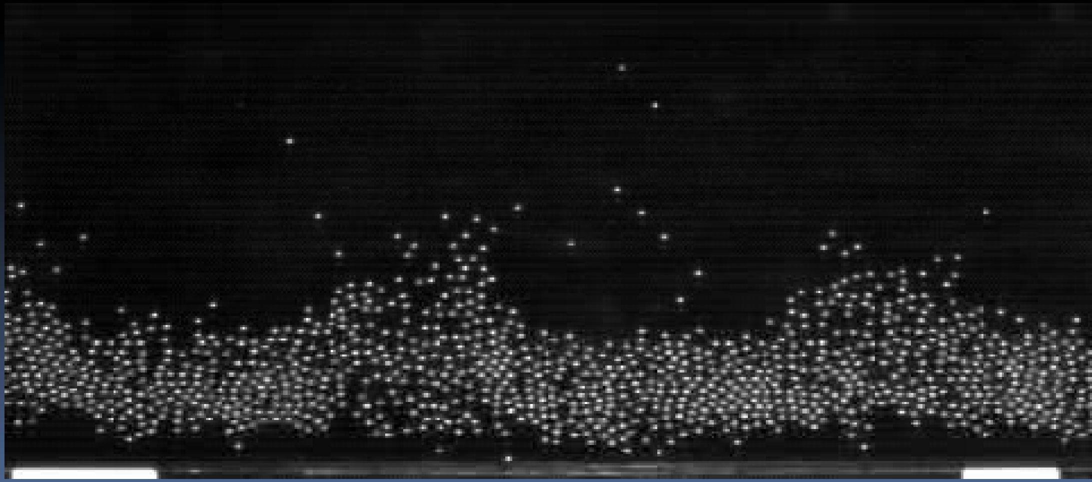
What is a granular *fluid*?

- When the granular matter is driven and energy is fed into the system (e.g., by shaking) such that the grains are not in constant contact with each other, the granular material is said to *fluidize*.





Minimal model of a granular gas: A gas of (smooth) *inelastic* hard spheres

Several circles
(Kandinsky, 1926)

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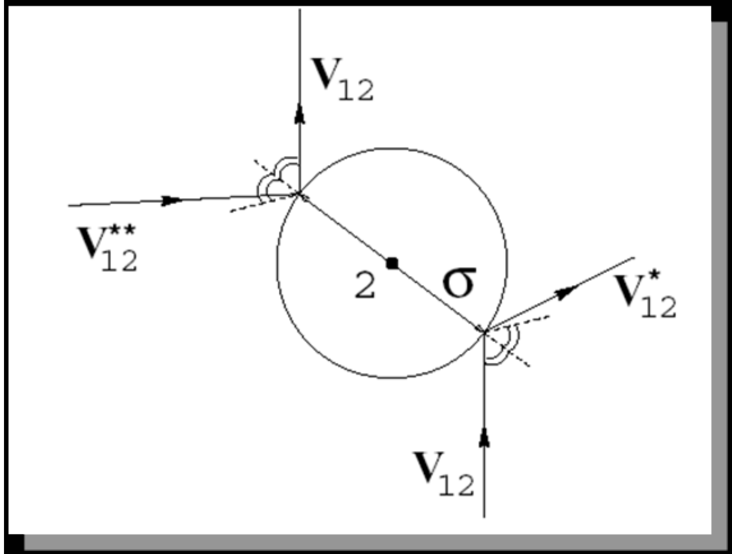


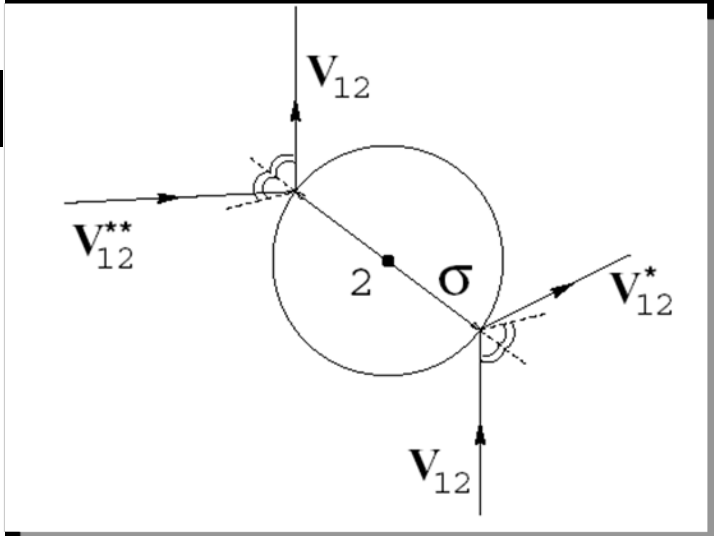
(After T.P.C. van Noije & M.H. Ernst)

- Mass m
- Diameter σ
- Coefficient of normal restitution α
- $\alpha=1$ for elastic collisions

Relative velocity

Direct collision: $\mathbf{v}_1^* = \mathbf{v}_1 - \frac{1+\alpha}{2}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2^* = \mathbf{v}_2 + \frac{1+\alpha}{2}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}$

Restituting collision: $\mathbf{v}_1^{**} = \mathbf{v}_1 - \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2^{**} = \mathbf{v}_2 + \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}$



Collisions conserve momentum, but not kinetic energy:

$$\Delta E = \frac{1}{2}m(v_1^{*2} + v_2^{*2} - v_1^2 - v_2^2) = -\frac{m}{2}(1-\alpha^2)(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})^2$$

"Granular" temperature: $T = \frac{m}{3}\langle(\mathbf{v}-\mathbf{u})^2\rangle, \quad \mathbf{u} = \langle\mathbf{v}\rangle$

$$\left.\frac{\partial T}{\partial t}\right|_{\text{coll}} = -\zeta T, \quad \zeta \propto 1-\alpha^2$$

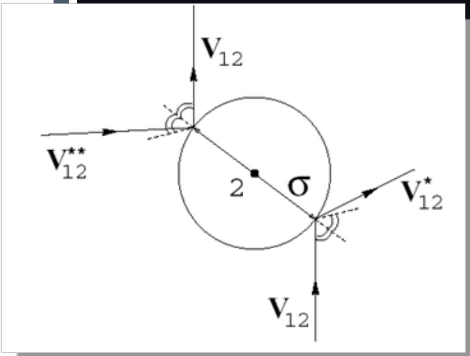
"Cooling" rate

Boltzmann equation (inelastic collisions)

$$\partial_t f + \mathbf{v}_1 \cdot \nabla f = J[f,f]$$

Collision operator

$$J[f,f] = \sigma^2 \int \mathrm{d}\mathbf{v}_2 \int \mathrm{d}\hat{\boldsymbol{\sigma}}\, \Theta(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}}) \times \left[\alpha^{-2} f(\mathbf{v}_1^{**}) f(\mathbf{v}_2^{**}) - f(\mathbf{v}_1) f(\mathbf{v}_2)\right]$$



$$\mathbf{v}_1^{**} = \mathbf{v}_1 - \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2^{**} = \mathbf{v}_2 + \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}}$$

Collisional Balance

$$\int \mathrm{d}\mathbf{v} \left\{ \begin{array}{c} 1 \\ \mathbf{v} \\ (\mathbf{v}-\mathbf{u})^2 \end{array} \right\} J[f,f] = \left\{ \begin{array}{c} 0 \\ \mathbf{0} \\ -\frac{3}{m}\zeta nT \end{array} \right\}$$

Cooling rate

- Conservation of mass
- Conservation of momentum
- *Energy sink*

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Our problem: Longitudinal flow

Hydrodynamic fields

$$\mathbf{u} = u_x \widehat{\mathbf{x}}, \quad u_x = ax$$

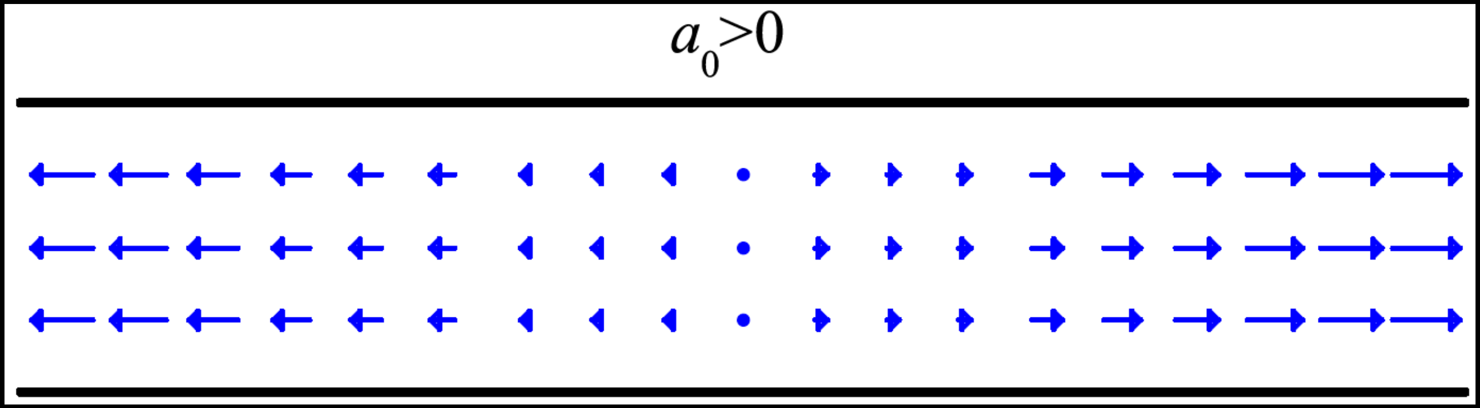
$$\nabla n = \nabla T = 0$$

Balance equations

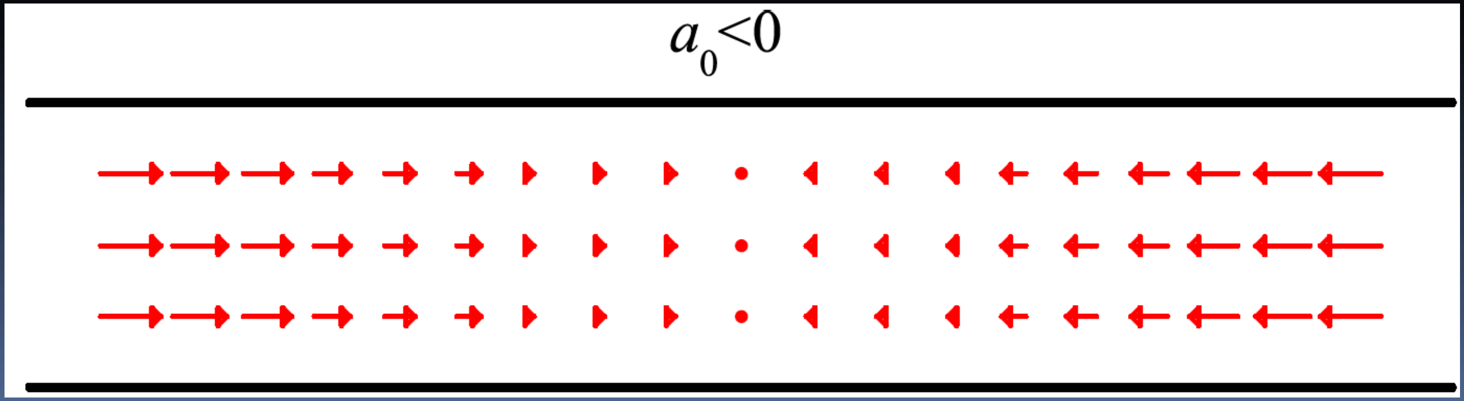
$$a(t) = \frac{a_0}{1 + a_0 t}, \quad n(t) = \frac{n_0}{a_0} a(t)$$

$$\partial_t T(t) = -\frac{2a_0}{3n_0} P_{xx}(t) - \zeta(t) T(t)$$

Expansion: Temperature monotonically decreases with time



Compression: Viscous heating competes with inelastic cooling
$\Rightarrow$ Stationary temperature



Mapping onto a uniform problem

Lagrangian frame:

$$f(\mathbf{r},\mathbf{v};t) \rightarrow f(\mathbf{V},t)$$

$$\mathbf{V} \equiv \mathbf{v} - \mathbf{u}(\mathbf{r},t)$$

The flow becomes equivalent to a uniform gas subject to a non-conservative force

$$\mathbf{F} = -ma_0 V_x \hat{\mathbf{x}}$$

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Relevant control parameter of the problem: reduced deformation rate

$$a^*(t) = \frac{a(t)}{\nu(t)}, \qquad |a^*| : \text{ Knudsen number}$$

$$\nu(t) \propto n(t)\sqrt{T(t)} : \text{effective collision frequency}$$

Relevant response function: reduced generalized viscosity

$$\frac{P_{xx}(t)}{p(t)} = \frac{T_x(t)}{T(t)} = 1 - \frac{4}{3}\eta^*(a^*,\alpha)a^*$$

Navier-Stokes: $\lim_{a^* \to 0} \eta^*(a^*,\alpha) = \frac{1}{1+\zeta^*/2}$

$$\zeta^* \equiv \frac{\zeta}{\nu} = \frac{5}{6}(1-\alpha)$$

Our two main objectives:

- Derive an ordinary differential equation for $\eta^*(a^*,\alpha)$.
- Investigate the convergence or divergence of the Chapman-Enskog expansion $\eta^*(a^*,\alpha)=\sum_{k=0}^{\infty} c_k(\alpha)\, a^{*k}$.

Model kinetic equation (BGK-like)

$$(\partial_t + \mathbf{v}\cdot\nabla)f = \underbrace{-\nu(f - f_0) + \frac{\zeta}{2}\partial_{\mathbf{v}}\cdot[(\mathbf{v}-\mathbf{u})f]}_{J[f,f]}$$

J. J. Brey, J. W. Dufty, & A. S., J. Stat. Phys. **97**, 281 (1999)

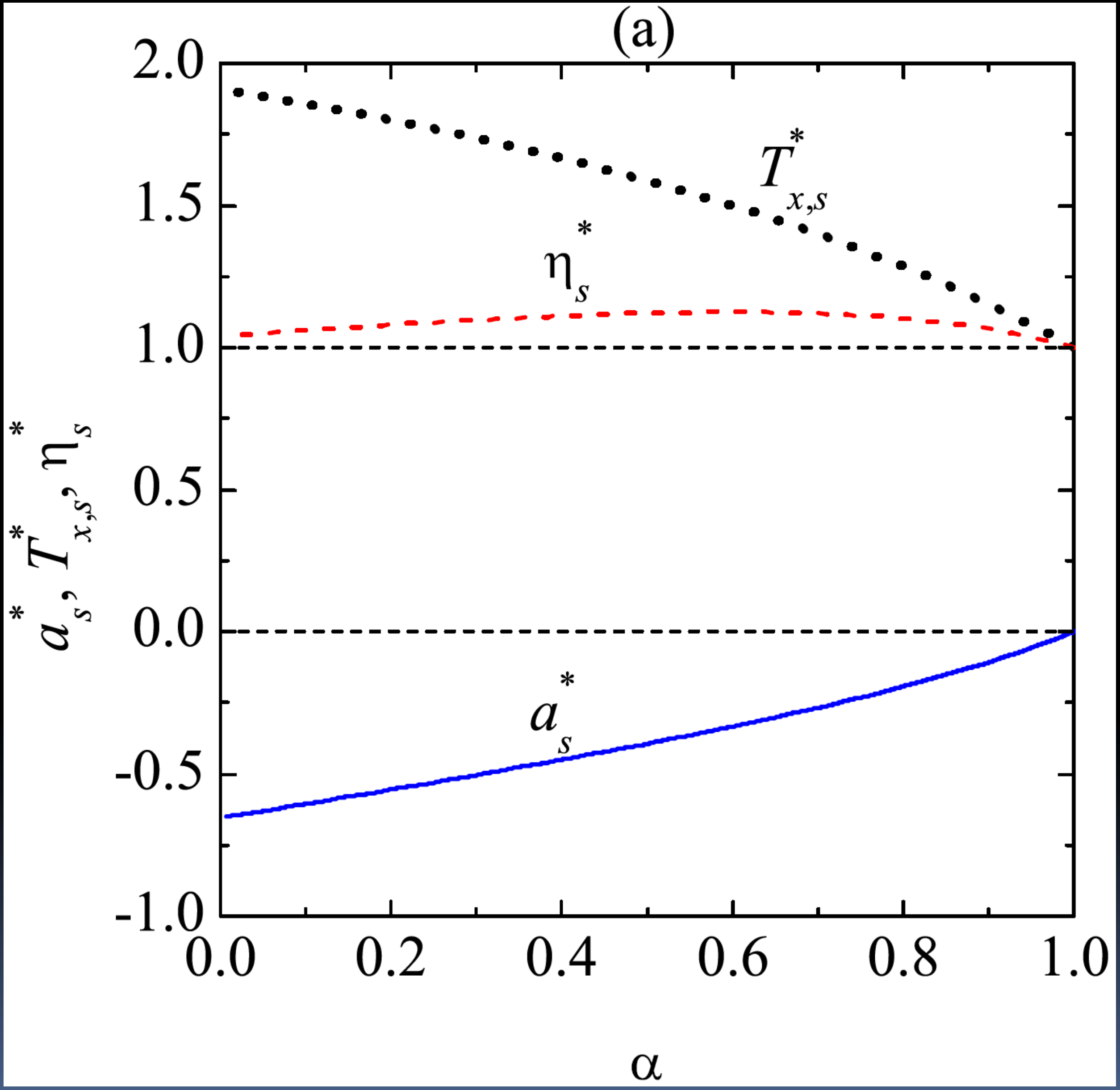
Moment equations

$$\partial_t T(t) = -\frac{2a_0}{3n_0} P_{xx}(t) - \zeta(t) T(t)$$

$$\partial_t P_{xx}(t) = \nu(t) p(t) - \left[\nu(t) + 3a(t) + \zeta(t)\right] P_{xx}(t)$$

If $a_0 < 0$, steady-state values of the reduced quantities:

$$a_s^*(\alpha) = -\frac{3}{2}\zeta^* \frac{1+\zeta^*}{1+3\zeta^*}, \qquad \eta_s^*(\alpha) = \frac{1+3\zeta^*}{(1+\zeta^*)^2}.$$

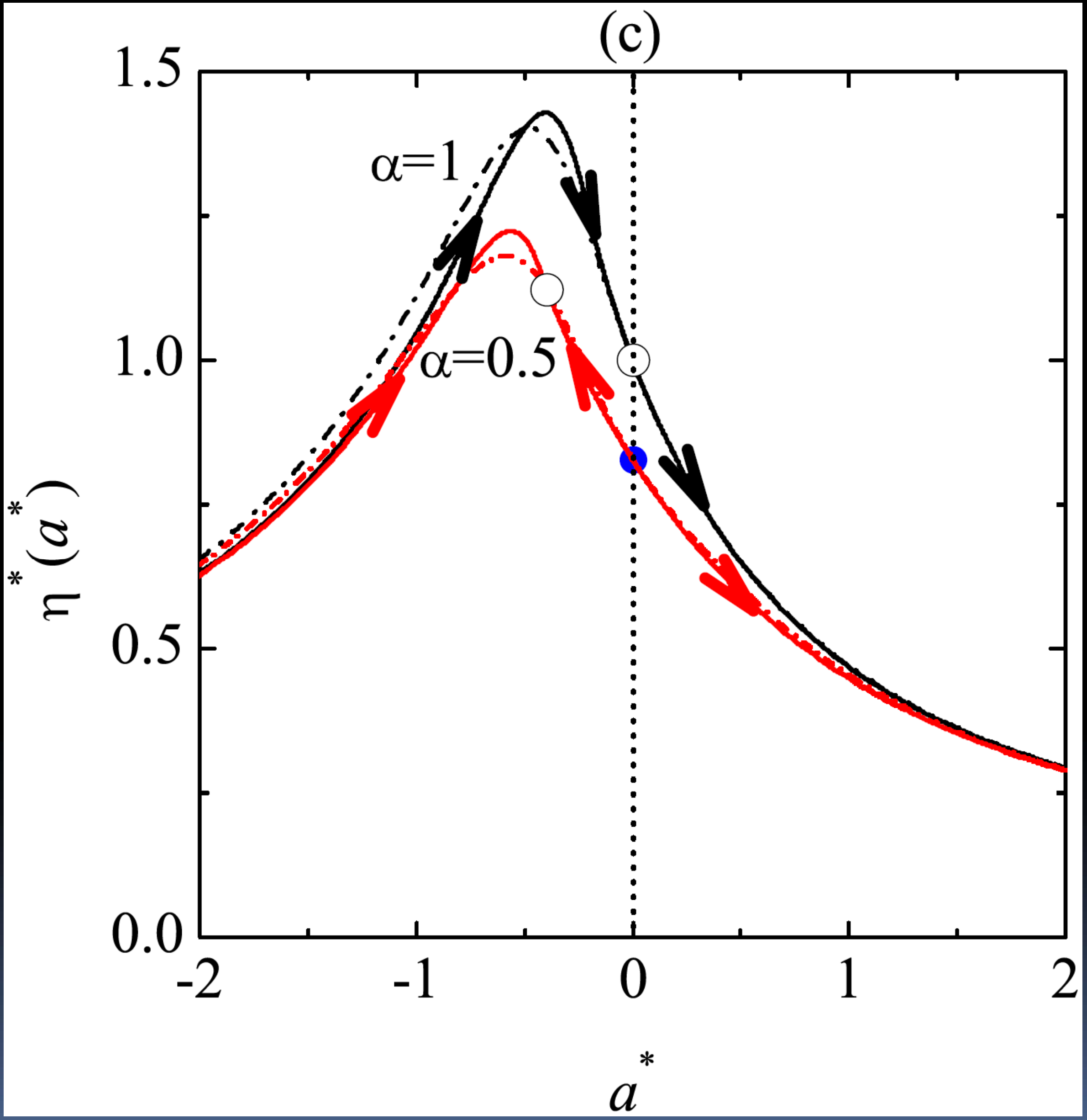

(a)
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
$T^*_{x,s}$
η^*_s
a^*_s
$a^*_s, T^*_{x,s}, \eta^*_s$
α

What about the whole function $\eta^*(a^*,\alpha)$?

- By eliminating time in favor of $a^*(t)$ one gets the ODE

$$\eta^* - \left(1-\frac{4}{3}\eta^* a^*\right)\left(1+\frac{2}{3}\eta^* a^*\right) + \frac{1}{2}\left[\zeta^* + \frac{2}{3}a^*\left(1-\frac{4}{3}\eta^* a^*\right)\right]\left(\eta^* + a^*\partial_{a^*}\eta^*\right) = 0$$

It must be solved numerically


(c)
1.5
1.0
0.5
0.0
-2
-1
0
1
2
α=1
α=0.5
η*(a*)
a*

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Chapman-Enskog expansion

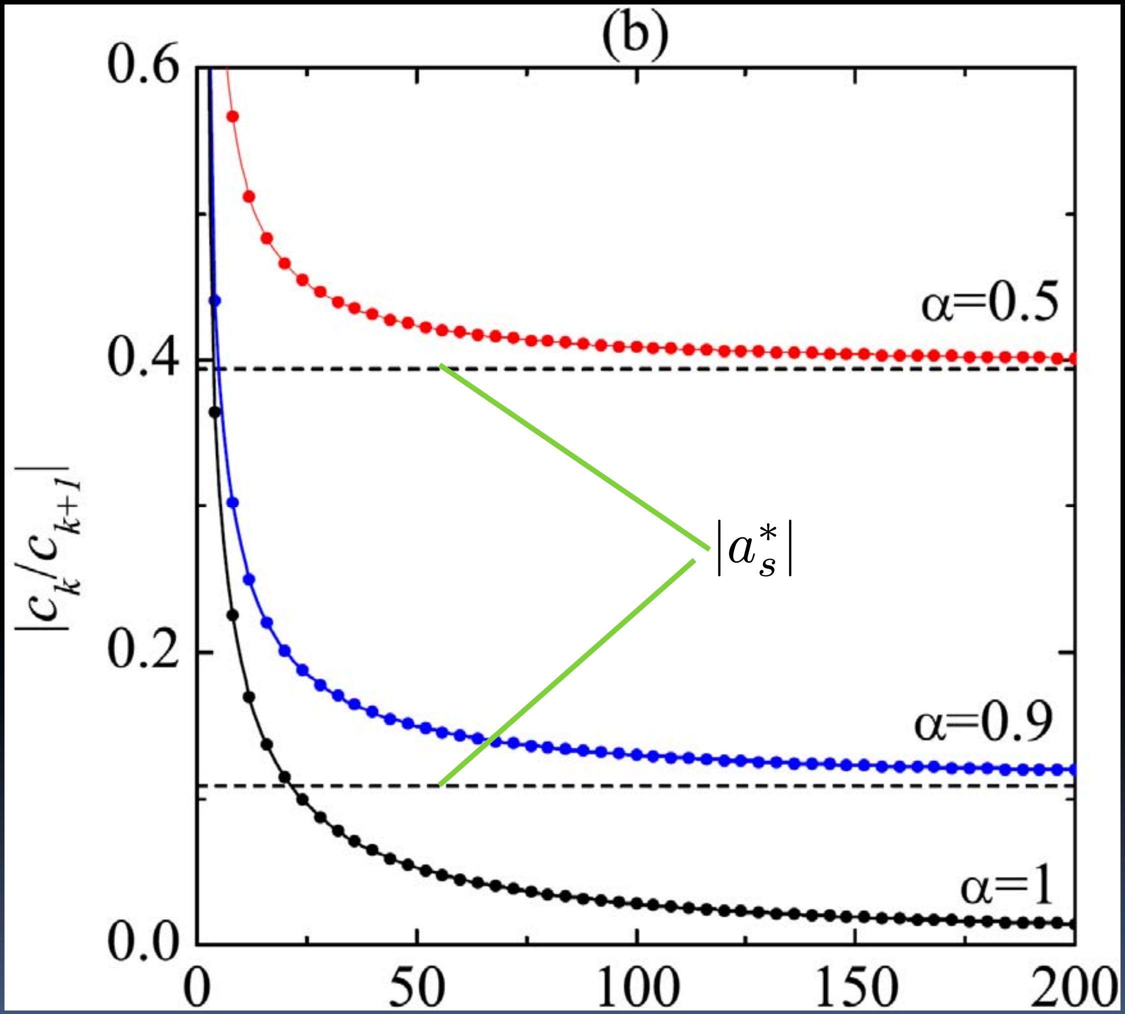
$$\eta^*(a^*, \alpha) = \sum_{k=0}^{\infty} c_k(\alpha) a^{*k},$$

$$c_0(\alpha) = \frac{1}{1 + \zeta^*/2}$$

Navier-Stokes

ODE $\Rightarrow$ recursion relation for $c_k(\alpha)$

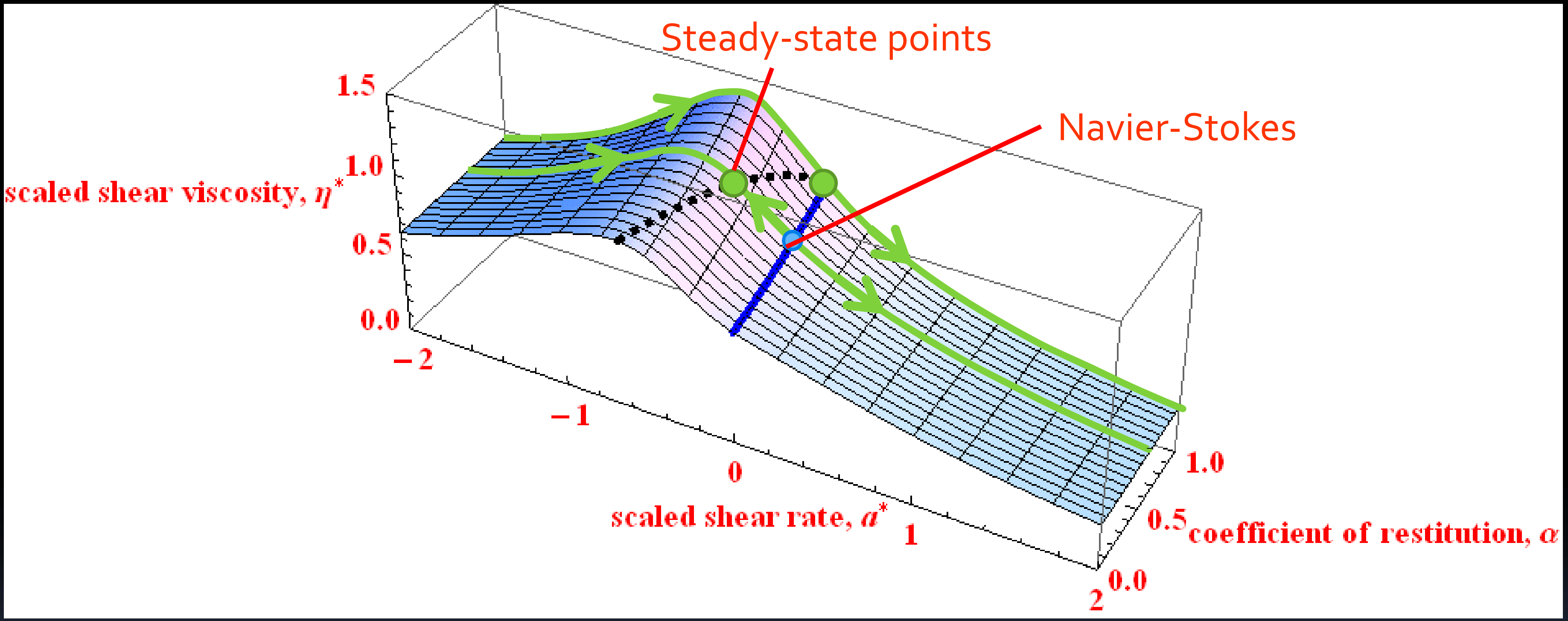
$$|c_k(\alpha)| \sim |a_s^*(\alpha)|^{-k}$$



Thus …

- The Chapman-Enskog series diverges for *elastic* collisions.
- But it converges for *inelastic* collisions!
- In fact, the stronger the inelasticity, the larger the radius of convergence.
- Can this paradoxical result be understood by physical arguments?

$\eta^*(a^*, \alpha)$



Steady-state points
Navier-Stokes
scaled shear viscosity, η^*
1.5
1.0
0.5
0.0
−2
−1
0
1
2
scaled shear rate, a^*
1.0
0.5
0.0
coefficient of restitution, α

$$\eta^*(a^*, \alpha) = \sum_{k=0}^{\infty} c_k(\alpha) {a^*}^k$$

- The reference homogeneous state ($a^*=0$) is an *attractor* of the evolution of $a^*(t)$ for elastic collisions ⇒ The CE expansion goes against the arrow of time ⇒ The CE series *diverges*.
- The state $a^*=0$ is a *repeller* of $a^*(t)$ for inelastic collisions ⇒ The CE expansion goes in favor of the arrow of time ⇒ The CE series *converges*.

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- The uniform longitudinal viscous flow is an unsteady compressible flow that, despite its apparent simplicity, constitutes a non-trivial playground for nonequilibrium statistical mechanics beyond the NS description.
- At a given value of α, the (scaled) nonlinear viscosity $\eta^*(a^*)$ defines a non-Newtonian rheological curve, the steady-state value $\eta^*_s=\eta^*(a^*_s)$ (where $a^*_s<0$) representing just one point.
- The Chapman-Enskog expansion of $\eta^*(a^*)$ diverges for ordinary gases (elastic collisions) but converges for granular gases (inelastic collisions).

Thank you for your attention!